Homework 9

Question 1

Radiation from Rotating Ring
An insulating circular ring (radius b) lies in the xy plane, centered at the origin. It carries a linear charge density $\lambda = \lambda_0 \sin\theta$, where λ_0 is constant and θ is the usual azimuthal angle. The ring is now set spinning at a constant angular velocity ω about the z axis. Calculate the power radiated.

Hint: The Poynting vector of a dipole is given by

$$\vec{S} = \frac{1}{4\pi c^3 r^2} \left| \hat{\mathbf{n}} \times \ddot{\vec{p}}\,(t_{\text{ret}}) \right|^2 \hat{\mathbf{n}}.$$

Note that you are not asked to take a time average of the radiation in this case.

Solution

The dipole moment of the ring when it is stationary is

$$\begin{aligned} p_0 &= \int \rho \vec{r}\,\mathrm{d}V = \int_0^{2\pi} \lambda_0 \sin\theta \left(b\cos\theta \hat{\mathbf{x}} + b\sin\theta \hat{\mathbf{y}} \right) b\,\mathrm{d}\theta \\ &= \lambda_0 b^2 \hat{\mathbf{y}} \int_0^{2\pi} \sin^2\theta\,\mathrm{d}\theta = \pi \lambda_0 b^2 \hat{\mathbf{y}}. \end{aligned}$$

When it is rotating, the dipole moment is

$$p\,(t) = p_0 \left[-\sin(\omega t)\,\hat{\mathbf{x}} + \cos(\omega t)\,\hat{\mathbf{y}} \right],$$

for $\vec{\omega} = \omega \hat{\mathbf{z}}$. The Poynting vector of the radiation is given by the angular distribution of the power,

$$|S_{\text{rad}}| = \frac{1}{r^2}\frac{dW}{d\Omega} = \frac{1}{4\pi c^3 r^2} \left| \hat{\mathbf{n}} \times \ddot{\vec{p}} \right|^2,$$

where $\hat{\mathbf{n}} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. Since $\ddot{\vec{p}} = -\omega^2\vec{p}$, we find the Poynting vector

$$\begin{aligned}
|S_{\text{rad}}| &= \frac{\omega^4}{4\pi c^3 r^2} |\hat{\mathbf{n}} \times \vec{p}|^2 \\
&= \frac{\omega^4 p_0^2}{4\pi c^3 r^2} \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ -\sin(\omega t_{\text{ret}}) & \cos(\omega t_{\text{ret}}) & 0 \end{vmatrix}^2 \\
&= \frac{\omega^4 p_0^2}{4\pi c^3 r^2} \left[\hat{\mathbf{x}}\left(-\cos(\omega t_{\text{ret}})\cos\theta\right) - \hat{\mathbf{y}}\left(\sin(\omega t_{\text{ret}})\cos\theta\right) + \hat{\mathbf{z}}\sin\theta\left(\underbrace{\cos\varphi\cos(\omega t_{\text{ret}}) + \sin\varphi\sin(\omega t_{\text{ret}})}_{=\cos(\omega t_{\text{ret}} - \varphi)}\right)\right]^2 \\
&= \frac{\omega^4 p_0^2}{4\pi c^3 r^2} \left[\cos^2\theta + \sin^2\theta\cos^2(\omega t_{\text{ret}} - \varphi)\right].
\end{aligned}$$

The Power is then

$$\begin{aligned}
P = \int \vec{S}\cdot \mathrm{d}\vec{a} &= \frac{\omega^4 p_0^2}{4\pi c^3 r^2} \int_0^{\pi}\int_0^{2\pi} r^2 \sin\theta \,\mathrm{d}\theta\,\mathrm{d}\varphi \left[\cos^2\theta + \sin^2\theta\cos^2(\omega t_{\text{ret}} - \varphi)\right] \\
&= \frac{\omega^4 p_0^2}{4\pi c^3} \int_0^{\pi}\int_0^{2\pi} \mathrm{d}\theta\,\mathrm{d}\varphi \left[\sin\theta\cos^2\theta + \sin^3\theta\cos^2(\omega t_{\text{ret}} - \varphi)\right] \\
&= \frac{2\omega^4 p_0^2}{3c^3} = \frac{2\pi^2\omega^4\lambda_0^2 b^4}{3c^3}.
\end{aligned}$$

Question 2

In this question you will find the radiation of an oscillating electric dipole.

Imagine two tiny metal spheres separated by a distance d and connected by a fine wire (see figure). At time t the charge on the upper sphere is $q(t)$ and on the lower sphere it is $-q(t)$. Suppose we drive the charge back and forth through the wire with angular frequency ω, such that

$$q(t) = q_0\cos(\omega t),$$

resulting in an oscillating electric dipole

$$\vec{p}(t) = p_0\cos(\omega t)\,\hat{\mathbf{z}} = q_0 d\cos(\omega t)\,\hat{\mathbf{z}}.$$

1. The retarded potential is

$$\Phi(t, \vec{r}) = \frac{1}{4\pi}\left[\frac{q_0\cos(\omega(t - r_+/c))}{r_+} - \frac{q_0\cos(\omega(t - r_-/c))}{r_-}\right].$$

   Write $r_\pm$ explicitly in terms of r, d and θ.

2. This *physical* dipole becomes a *perfect* dipole when the separation distance is very small.

Approximate $d \ll r$ (but $d > 0$!) and show that to first order in d,

$$\cos\left(\omega\left(t - r_{\pm}/c\right)\right) \cong \cos\left(\omega\left(t - r/c\right)\right)\cos\left(\frac{\omega d}{2c}\cos\theta\right) \mp \sin\left(\omega\left(t - r/c\right)\right)\sin\left(\frac{\omega d}{2c}\cos\theta\right).$$

3. In the perfect dipole limit, we further have

$$d \ll \frac{2\pi c}{\omega} = \lambda.$$

Under this assumption, show that the potential of an oscillating perfect dipole is

$$\Phi\left(t, r, \theta\right) \cong \frac{p_0 \cos\theta}{4\pi r}\left[-\frac{\omega}{c}\sin\left(\omega\left(t - r/c\right)\right) + \frac{1}{r}\cos\left(\omega\left(t - r/c\right)\right)\right]. \tag{1}$$

4. To find the radiation of the dipole, not every piece of the potential is useful; Only the fields which survive large at distances for the source, in the *radiation zone*

$$r \gg \frac{c}{\omega} = \frac{\lambda}{2\pi},$$

contribute to the radiation. Write the scalar potential in this case.

5. The vector potential is determined by the current flowing in the wire,

$$I\left(t\right) = \frac{\mathrm{d}q}{\mathrm{d}t}\hat{\mathbf{z}} = -q_0\omega\sin\left(\omega t\right)\hat{\mathbf{z}}.$$

Show that the vector potential can be similarly written in the form

$$\vec{A}\left(t, \vec{r}\right) = -\frac{p_0\omega}{cr}\sin\left[\omega\left(t - r/c\right)\right]\mathrm{d}z\hat{\mathbf{z}}.$$

Hint: Approximate the integral on the vector potential to first order in d, such that the integration is replaced by a factor of d and the integrand is evaluated at the center.

6. Find the electric and magnetic fields under the approximation $r \gg \lambda$.

7. Find the time average Poynting vector, where the average is taken over a complete cycle.

8. Using the relation between intensity and power,

$$\left\langle P\right\rangle_T = \int \left\langle \vec{S}\right\rangle_T \cdot \mathrm{d}\vec{A},$$

where A is the surface over a sphere, find the radiation power. Solution

9. According to the figure,

$$r_{\pm} = \sqrt{r^2 \mp rd\cos\theta + (d/2)^2}.$$

10. In the limit of very small separation, we have

$$r_{\pm} \cong r\left(1 \mp \frac{d}{2r}\cos\theta\right),$$

and thus

$$\frac{1}{r_{\pm}} \cong \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right).$$

The cosine becomes

$$\begin{aligned}\cos\left(\omega\left(t - r_{\pm}/c\right)\right) &\cong \cos\left[\omega\left(t - r/c\right) \pm \frac{\omega d}{2c}\cos\theta\right] \\ &= \cos\left(\omega\left(t - r/c\right)\right)\cos\left(\frac{\omega d}{2c}\cos\theta\right) \mp \sin\left(\omega\left(t - r/c\right)\right)\sin\left(\frac{\omega d}{2c}\cos\theta\right).\end{aligned}$$

11. The approximation $d \ll c/\omega$ gives

$$x = \frac{\omega d}{2c}\cos\theta \ll 1,$$

which means we can approximate

$$\cos(x) \approx 1,\ \sin(x) \approx x,$$

and obtain

$$\cos\left(\omega\left(t - r_{\pm}/c\right)\right) \cong \cos\left(\omega\left(t - r/c\right)\right) \mp \frac{\omega d}{2c}\cos\theta\sin\left(\omega\left(t - r/c\right)\right),$$

and the potential becomes

$$\begin{aligned}\Phi(t,r,\theta) &\cong \frac{q_0}{4\pi}\left[\frac{\left[\cos\left(\omega\left(t - r/c\right)\right) - \frac{\omega d}{2c}\cos\theta\sin\left(\omega\left(t - r/c\right)\right)\right]}{r_+} - \frac{\left[\cos\left(\omega\left(t - r/c\right)\right) + \frac{\omega d}{2c}\cos\theta\sin\left(\omega\left(t - r/c\right)\right)\right]}{r_-}\right] \\ &\cong \frac{q_0}{4\pi r}\left\{\left(1 + \frac{d}{2r}\cos\theta\right)\left[\cos\left(\omega\left(t - r/c\right)\right) - \frac{\omega d}{2c}\cos\theta\sin\left(\omega\left(t - r/c\right)\right)\right]\right. \\ &\left. - \left(1 - \frac{d}{2r}\cos\theta\right)\left[\cos\left(\omega\left(t - r/c\right)\right) + \frac{\omega d}{2c}\cos\theta\sin\left(\omega\left(t - r/c\right)\right)\right]\right\} \\ &\cong \frac{p_0\cos\theta}{4\pi r}\left[-\frac{\omega}{c}\sin\left(\omega\left(t - r/c\right)\right) + \frac{1}{r}\cos\left(\omega\left(t - r/c\right)\right)\right],\end{aligned}$$

where in the last line we discarded the terms proportional to d^2 as they are very small.

12. In the radiation zone, only the sine term contributes since $\omega/c \gg r^{-1}$ and we find

$$\Phi\left(t, r, \theta\right) \cong -\frac{p_0 \omega \cos\theta}{4\pi c r} \sin\left[\omega\left(t - r/c\right)\right].$$

13. We use the integral form of $\vec{A}$,

$$\vec{A}\left(t, \vec{r}\right) = \frac{1}{c} \iiint \frac{\vec{J}\left(\vec{r}', t_r\right)}{\left|\vec{r} - \vec{r}'\right|} \mathrm{d}^3 r',$$

where

$$\vec{J}\left(\vec{r}', t_r\right) \mathrm{d}^3 r' = I\left(t_r\right) dz\hat{\mathbf{z}}.$$

Therefore we find the vector potential

$$\vec{A}\left(t, \vec{r}\right) = \frac{1}{c} \int_{-d/2}^{d/2} \frac{-q_0 \omega \sin\left(\omega t_r\right)}{\left|\vec{r} - z'\hat{\mathbf{z}}\right|} dz\hat{\mathbf{z}}.$$

Since d is very small, we can approximate the integral by the length of the interval times the integrand evaluated at its center, $z = 0$:

$$\vec{A}\left(t, \vec{r}\right) = d\left(\frac{1}{c} \frac{-q_0 \omega \sin\left(\omega t_r\left(z = 0\right)\right)}{\left|\vec{r} - 0\hat{\mathbf{z}}\right|}\right)\hat{\mathbf{z}} = -\frac{p_0 \omega}{c r} \sin\left[\omega\left(t - r/c\right)\right]\hat{\mathbf{z}}.$$

14. The fields are

$$\begin{aligned} \vec{E} &= -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}, \\ \vec{B} &= \vec{\nabla} \times \vec{A}. \end{aligned}$$

We have

$$\begin{aligned} \vec{\nabla}\Phi &= \frac{\partial \Phi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial \Phi}{\partial r}\hat{\boldsymbol{\theta}} \\ &= \frac{p_0 \omega}{4\pi c}\left\{\cos\theta\left(\frac{1}{r}\sin\left[\omega\left(t - r/c\right)\right] + \frac{\omega}{r c}\cos\left[\omega\left(t - r/c\right)\right]\right)\hat{\mathbf{r}} + \frac{\sin\theta}{r^2}\sin\left[\omega\left(t - r/c\right)\right]\hat{\boldsymbol{\theta}}\right\} \\ &\overset{r \gg \lambda}{\cong} \frac{p_0 \omega^2}{4\pi c^2}\left(\frac{\cos\theta}{r}\right)\cos\left[\omega\left(t - r/c\right)\right]\hat{\mathbf{r}}, \end{aligned}$$

and

$$\frac{\partial \vec{A}}{\partial t} = -\frac{p_0 \omega^2}{c r}\cos\left[\omega\left(t - r/c\right)\right]\hat{\mathbf{z}} = -\frac{p_0 \omega^2}{c r}\cos\left[\omega\left(t - r/c\right)\right]\left(\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}\right),$$

which gives the electric field

$$\vec{E} = -\frac{p_0\omega^2}{c}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t - r/c\right)\right]\hat{\boldsymbol{\theta}}.$$

For the magnetic field, we compute

$$\begin{aligned}
\vec{\nabla}\times\vec{A} &= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial\theta}\right]\hat{\boldsymbol{\varphi}} \\
&= -\frac{p_0\omega}{cr}\left\{\frac{\omega}{c}\sin\theta\cos\left[\omega\left(t - r/c\right)\right] + \frac{\sin\theta}{r}\sin\left[\omega\left(t - r/c\right)\right]\right\}\hat{\boldsymbol{\varphi}} \\
&\overset{r\gg\lambda}{\cong} -\frac{p_0\omega^2}{c^2}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t - r/c\right)\right]\hat{\boldsymbol{\varphi}},
\end{aligned}$$

and we find

$$\vec{B} = -\frac{p_0\omega^2}{c^2}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t - r/c\right)\right]\hat{\boldsymbol{\varphi}}.$$

15. The Poynting vector for the oscillating dipole is

$$\vec{S} = \frac{c}{4\pi}\left(\vec{E}\times\vec{B}\right) = \frac{1}{c^2}\left\{p_0\omega^2\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t - r/c\right)\right]\right\}^2\hat{\mathbf{r}}.$$

The intensity, averaged over a single cycle, where

$$\left\langle\cos^2\left[\omega\left(t - r/c\right)\right]\right\rangle_T = \frac{1}{2},$$

is simply

$$\left\langle\vec{S}\right\rangle_T = \frac{p_0^2\omega^4}{2c^2r^2}\sin^2\theta\hat{\mathbf{r}}.$$

16. We use the fact that $\mathrm{d}A = r^2\,\mathrm{d}\Omega$ to obtain that

$$\langle P\rangle_T = \int \mathrm{d}\Omega\left\langle\vec{S}\right\rangle_T r^2\cdot\hat{\mathbf{r}} = \frac{p_0^2\omega^4}{3c^2}.$$

Question 3

Radiation for Bohr's Atom
In Bohr's theory of hydrogen, the electron in its ground state was supposed to travel in a circle of radius 5×10^{-11}, held in orbit by the Coulomb attraction of the proton.
According to classical electrodynamics, this electron should radiate, and hence spiral in to the nucleus. Show that $v \ll c$ for most of the trip (so you can use the Larmor formula), and calculate the lifespan of Bohr's atom. (Assume each revolution is essentially circular.)

Solution

$F = \frac{1}{4\pi\epsilon_0}\frac{q^2}{r^2} = ma = m\frac{v^2}{r} \Rightarrow v = \sqrt{\frac{1}{4\pi\epsilon_0}\frac{q^2}{mr}}$. At the beginning ($r_0 = 0.5\,\text{Å}$),

$$\frac{v}{c} = \left[\frac{(1.6\times10^{-19})^2}{4\pi(8.85\times10^{-12})(9.11\times10^{-31})(5\times10^{-11})}\right]^{-1/2}\frac{1}{3\times10^8} = 0.0075,$$

and when the radius is one hundredth of this v/c is only 10 times greater (0.075), so for *most* of the trip the velocity is safely nonrelativistic.

¿From the Larmor formula, $P = \frac{\mu_0 q^2}{6\pi c}\left(\frac{v^2}{r}\right)^2 = \frac{\mu_0 q^2}{6\pi c}\left(\frac{1}{4\pi\epsilon_0}\frac{q^2}{mr^2}\right)^2$ (since $a = v^2/r$), and $P = -dU/dt$, where U is the (total) energy of the electron:

$$U = U_{\text{kin}} + U_{\text{pot}} = \frac{1}{2}mv^2 - \frac{1}{4\pi\epsilon_0}\frac{q^2}{r} = \frac{1}{2}\left(\frac{1}{4\pi\epsilon_0}\frac{q^2}{r}\right) - \frac{1}{4\pi\epsilon_0}\frac{q^2}{r} = -\frac{1}{8\pi\epsilon_0}\frac{q^2}{r}.$$

So $-\frac{dU}{dt} = -\frac{1}{8\pi\epsilon_0}\frac{q^2}{r^2}\frac{dr}{dt} = P = \frac{q^2}{6\pi\epsilon_0 c^3}\left(\frac{1}{4\pi\epsilon_0}\frac{q^2}{mr^2}\right)^2$, and hence $\frac{dr}{dt} = -\frac{1}{3c}\left(\frac{q^2}{2\pi\epsilon_0 mc}\right)^2\frac{1}{r^2}$, or

$dt = -3c\left(\frac{2\pi\epsilon_0 mc}{q^2}\right)^2 r^2\,dr \Rightarrow t = -3c\left(\frac{2\pi\epsilon_0 mc}{q^2}\right)^2\int_{r_0}^{0} r^2\,dr = \boxed{c\left(\frac{2\pi\epsilon_0 mc}{q^2}\right)^2 r_0^3}$

$= (3\times10^8)\left[\frac{2\pi(8.85\times10^{-12})(9.11\times10^{-31})(3\times10^8)}{(1.6\times10^{-19})^2}\right]^2(5\times10^{-11})^3 = \boxed{1.3\times10^{-11}\,\text{s.}}$ (Not very long!)

Question 4

1. Suppose an electron decelerates at a constant rate from some initial velocity $v_0 \ll c$ down to zero. Calculate what fraction of the initial kinetic energy is lost to radiation (the rest is absorbed by whatever mechanism keeps the acceleration constant).

2. To get a sense of the numbers involved, suppose that the initial velocity is thermal ($\sim 10^5\,[m/s]$) and that the distance the electron travels until it stops is 30 ångström ($10^{-10}[m]$). What can you conclude about radiation losses for the electrons in an ordinary conductor?

Solution

(a) $P = \dfrac{\mu_0 q^2 a^2}{6\pi c}$, and the time it takes to come to rest is $t = v_0/a$, so the energy radiated is $U_{\text{rad}} = Pt = \dfrac{\mu_0 q^2 a^2}{6\pi c}\dfrac{v_0}{a}$. The initial kinetic energy was $U_{\text{kin}} = \frac{1}{2}mv_0^2$, so the fraction radiated is $f = \dfrac{U_{\text{rad}}}{U_{\text{kin}}} = \boxed{\dfrac{\mu_0 q^2 a}{3\pi m v_0 c}}$.

(b) $d = \dfrac{1}{2}at^2 = \dfrac{1}{2}a\dfrac{v_0^2}{a^2} = \dfrac{v_0^2}{2a}$, so $a = \dfrac{v_0^2}{2d}$. Then

$$f = \frac{\mu_0 q^2}{3\pi m v_0 c}\frac{v_0^2}{2d} = \frac{\mu_0 q^2 v_0}{6\pi m c d} = \frac{(4\pi \times 10^{-7})(1.6 \times 10^{-19})^2(10^5)}{6\pi(9.11 \times 10^{-31})(3 \times 10^8)(3 \times 10^{-9})} = \boxed{2 \times 10^{-10}}.$$

The losses due to radiation in this case are very small, and we conclude that losses due to collisions in an ordinary conductor should be negligible.