Class Exercise 11 - Multipole expansion of radiation

Problem 1

A ball of radius R is uniformly charged with charge density ρ and is rotating around its axis with angular velocity $\Omega(t) = \Omega_0 e^{-t^2/\tau^2}$, where Ω_0, τ are constants. Find the angular distribution, total energy and spectral distribution of the radiation.

Solution

We notice that when $t \to \pm\infty$, the angular velocity vanishes. Since the relation between a rotating and non-rotating dipole is a rotation matrix, we will compute the dipole moment of static sphere:

$$\vec{p} = \int \rho \vec{r'} \, \mathrm{d}^3 r' = \rho \iiint \begin{pmatrix} r' \sin\theta \cos\varphi \\ r' \sin\theta \sin\varphi \\ r' \cos\theta \end{pmatrix} r'^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi \, \mathrm{d}r' = 0.$$

As expected, the dipole moment is zero. This fact is unchanged by the rotation of the sphere, since the charge distribution remains uniform despite the rotation. We thus find no dipole radiation, so we will search for magnetic dipole radiation next.

We want to compute

$$\vec{m} = \frac{1}{2c} \int \vec{r'} \times \vec{J}(\vec{r'}) \, \mathrm{d}^3 r',$$

so we begin with finding the current. Each layer of the ball is rotating with velocity

$$\vec{v} = \vec{\Omega} \times \vec{r} = \Omega \hat{\mathbf{z}} \times \vec{r} = \Omega r \sin\theta \left(-\sin\varphi \hat{\mathbf{x}} + \cos\varphi \hat{\mathbf{y}}\right),$$

and thus we have

$$\begin{aligned}\vec{r}\times\vec{J} &= r\hat{\mathbf{r}}\times\left(\rho\Omega r\sin\theta\left(-\sin\varphi\hat{\mathbf{x}}+\cos\varphi\hat{\mathbf{y}}\right)\right)\\ &= \rho\Omega r^2\sin\theta\left(-\cos\theta\cos\varphi\hat{\mathbf{x}}-\cos\theta\sin\varphi\hat{\mathbf{y}}+\sin\theta\hat{\mathbf{z}}\right)\\ &= -\rho\Omega r^2\sin\theta\hat{\boldsymbol{\theta}}.\end{aligned}$$

Integration over φ yields zero for the x, y components and we find the magnetic dipole in the z direction

$$\begin{aligned}m_z &= \frac{\rho\Omega}{2c}\int_0^R\int_0^{2\pi}\int_0^{\pi} r^2\sin^2\theta r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi\,\mathrm{d}r\\ &= \frac{4\pi\rho\Omega\left(t\right)R^5}{15c}\equiv m_0 e^{-t^2/\tau^2},\end{aligned}$$

where we denote $m_0\equiv 4\pi\rho\Omega_0 R^5/\left(15c\right)$. The magnetic dipole radiation depends on $\ddot{\vec{m}}$, so we calculate it:

$$\begin{aligned}\dot{\vec{m}} &= -\frac{2t}{\tau^2}m_0 e^{-t^2/\tau^2}\hat{\mathbf{z}},\\ \ddot{\vec{m}} &= \left[-\frac{2}{\tau^2}+\left(\frac{2t}{\tau^2}\right)^2\right]m_0 e^{-t^2/\tau^2}\hat{\mathbf{z}},\end{aligned}$$

and so we find

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega}=\frac{\left|\ddot{\vec{m}}\times\hat{\mathbf{n}}\right|^2}{4\pi c^3}=\frac{m_0^2\left[-\frac{2}{\tau^2}+\left(\frac{2t}{\tau^2}\right)^2\right]^2 e^{-2t^2/\tau^2}\sin^2\theta}{4\pi c^3},$$

where we used the fact that

$$\left|\hat{\mathbf{z}}\times\hat{\mathbf{n}}\right|^2=\left|\hat{\mathbf{z}}\right|^2\left|\hat{\mathbf{n}}\right|^2\sin^2\theta,$$

since $\hat{\mathbf{n}}=\hat{\mathbf{r}}$ which has a θ angle with the z axis. The total radiation power is therefore

$$P=\int\frac{\mathrm{d}P}{\mathrm{d}\Omega}\,\mathrm{d}\Omega=\frac{2}{3c^3}m_0^2\left[-\frac{2}{\tau^2}+\left(\frac{2t}{\tau^2}\right)^2\right]^2 e^{-2t^2/\tau^2}.$$

Note that there is no-need to take the time average here - there are no oscillating functions involved.

The total energy lost by the radiation is then

$$E = \int_{-\infty}^{\infty} P\,\mathrm{d}t = \frac{2m_0^2}{3c^3}\int_{-\infty}^{\infty}\left[-\frac{2}{\tau^2}+\left(\frac{2t}{\tau^2}\right)^2\right]^2 e^{-2t^2/\tau^2}\,\mathrm{d}t$$
$$= \frac{2m_0^2}{3c^3}\int_{-\infty}^{\infty}\left[\frac{4}{\tau^4}-\frac{16t^2}{\tau^6}+\frac{16t^4}{\tau^8}\right] e^{-2t^2/\tau^2}\,\mathrm{d}t$$
$$= \frac{8m_0^2}{3c^3\tau^4}\int_{-\infty}^{\infty}\left[1-4\left(\frac{t}{\tau}\right)^2+4\left(\frac{t}{\tau}\right)^4\right] e^{-2t^2/\tau^2}\,\mathrm{d}t.$$

We change variables to $x \equiv \sqrt{2}t/\tau$ and obtain

$$E = \frac{8m_0^2}{3\sqrt{2}c^3\tau^3}\int_{-\infty}^{\infty}\left[1-2x^2+x^4\right] e^{-x^2}\,\mathrm{d}x.$$

We compute each term separately,

$$\int_{-\infty}^{\infty} x^2 e^{-x^2}\,\mathrm{d}x = \int_{-\infty}^{\infty}\left(-\frac{\partial}{\partial\alpha}e^{-\alpha x^2}\right)\bigg|_{\alpha=1}\mathrm{d}x$$
$$= -\frac{\partial}{\partial\alpha}\left(\int_{-\infty}^{\infty} e^{-\alpha x^2}\,\mathrm{d}x\right)\bigg|_{\alpha=1}$$
$$= -\frac{\partial}{\partial\alpha}\left(\sqrt{\frac{\pi}{\alpha}}\right)\bigg|_{\alpha=1}$$
$$= \frac{\sqrt{\pi}}{2\alpha^{3/2}}\bigg|_{\alpha=1} = \frac{\sqrt{\pi}}{2},$$

$$\int_{-\infty}^{\infty} x^4 e^{-x^2}\,\mathrm{d}x = \frac{\partial}{\partial\alpha^2}\left(\int_{-\infty}^{\infty} e^{-\alpha x^2}\,\mathrm{d}x\right)\bigg|_{\alpha=1}$$
$$= \frac{\partial}{\partial\alpha}\left(\frac{\sqrt{\pi}}{2\alpha^{3/2}}\right)\bigg|_{\alpha=1}$$
$$= \frac{3}{2}\frac{\sqrt{\pi}}{2\alpha^{5/2}}\bigg|_{\alpha=1} = \frac{3\sqrt{\pi}}{4},$$

$$\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}.$$

We therefore find the total energy

$$E = \frac{8\sqrt{\pi}m_0^2}{3\sqrt{2}c^3\tau^3}\left[1-2\frac{1}{2}+\frac{3}{4}\right] = \frac{\sqrt{2\pi}m_0^2}{c^3\tau^3} = \frac{16\pi^2\sqrt{2\pi}\rho^2\Omega^2R^{10}}{225c^5\tau^3}.$$

To obtain the spectral distribution of the radiation we define

$$E = \int_0^\infty u_\omega \, \mathrm{d}\omega,$$

which essentially quantifies the energy per unit frequency. Starting with Larmor's formula

$$P(t) = \frac{2\left|\ddot{m}\right|^2}{3c^3},$$

we decompose m into its Fourier modes

$$m(t) = \int_{-\infty}^{\infty} m_\omega e^{-i\omega t} \, \mathrm{d}\omega,$$

and using Parseval's theorem we find

$$\begin{aligned} E &= \int_{-\infty}^{\infty} P(t) \, \mathrm{d}t = \int_{-\infty}^{\infty} \frac{2\left|\ddot{m}\right|^2}{3c^3} \, \mathrm{d}t \\ &= \frac{2}{3c^3} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|\ddot{m}_\omega\right|^2 \mathrm{d}\omega \\ &= \frac{2}{3\pi c^3} \int_0^\infty \left|\ddot{m}_\omega\right|^2 \mathrm{d}\omega \\ &= \frac{2}{3\pi c^3} \int_0^\infty \omega^4 \left|m_\omega\right|^2 \mathrm{d}\omega \equiv \int_0^\infty u_\omega \, \mathrm{d}\omega. \end{aligned}$$

To compute the integral, we notice that

$$\begin{aligned} m_\omega &= \int_{-\infty}^{\infty} m(t) \, e^{i\omega t} \, \mathrm{d}t \\ &= \int_{-\infty}^{\infty} m_0 \exp\left[-\frac{t^2}{\tau^2} + i\omega t\right] \mathrm{d}t \\ &= m_0 e^{-\omega^2\tau^2/4} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{\tau^2}\left(t - \frac{i\omega\tau^2}{2}\right)^2\right] \mathrm{d}t \\ &= m_0 e^{-\omega^2\tau^2/4} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{\tau^2}\widetilde{t}^2\right] \mathrm{d}\widetilde{t} \\ &= m_0 e^{-\omega^2\tau^2/4} \sqrt{\pi}\tau. \end{aligned}$$

Therefore, we find that

$$u_\omega = \frac{2\tau^2\omega^4}{3c^3} m_0^2 e^{-\omega^2\tau^2/2}.$$

Problem 2

Consider a ball with a radius which changes in time according to

$$R(t) = R_0 + a\sin(\omega t)\cos(2\theta),$$

for $a \ll R$. The ball is uniformly charged with charge density ρ. Find the angular distribution and total power of the radiation.

Solution

We begin by computing the electric dipole,

$$\begin{aligned}
\vec{p} = \int \rho \vec{r'}\, \mathrm{d}^3 r' &= \rho \iiint \begin{pmatrix} r' \sin\theta\cos\varphi \\ r'\sin\theta\sin\varphi \\ r'\cos\theta \end{pmatrix} r'^2 \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\varphi\, \mathrm{d}r' \\
&= \hat{\mathbf{z}}\rho \int_0^{\pi}\int_0^{2\pi} \frac{R^4(t)}{4}\cos\theta\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
&= \hat{\mathbf{z}}\frac{\pi\rho}{2}\int_{-1}^{1}\Bigg(R_0 + a\sin(\omega t)\underbrace{\cos(2\theta)}_{2\cos^2\theta - 1}\Bigg)^4 \cos\theta\,\mathrm{d}(\cos\theta) \\
&= \hat{\mathbf{z}}\frac{\pi\rho R_0^2}{2}\int_{-1}^{1}\underbrace{\left(1+\frac{a}{R_0}\sin(\omega t)\left(2x^2-1\right)\right)^4}_{\text{symmetric}}\overbrace{x}^{\text{anti-symmetric}}\,\mathrm{d}x = 0.
\end{aligned}$$

We find no electric dipole radiation. There is also no magnetic dipole radiation since $\vec{J} = 0$ and thus $\vec{m} = 0$. We therefore have to compute the next order contribution of the quadrupole moment,

$$Q_{ij} \equiv \int \rho\left(3r'_i r'_j - \delta_{ij} r'^2\right) d^3 r'.$$

We begin with the diagonal elements,

$$
\begin{aligned}
Q_{xx} &= \rho \int_0^{2\pi} \int_{-1}^{1} \int_0^{R(t)} \left(3x'^2 - r'^2\right) r'^2 \,\mathrm{d}r' \,\mathrm{d}\left(\cos\theta\right) \mathrm{d}\varphi \\
&= \rho \int_0^{2\pi} \int_{-1}^{1} \int_0^{R(t)} \left(3r'^2 \sin^2\theta \cos^2\varphi - r'^2\right) r'^2 \,\mathrm{d}r' \,\mathrm{d}\left(\cos\theta\right) \mathrm{d}\varphi \\
&= \rho \int_0^{2\pi} \int_{-1}^{1} \left(3\sin^2\theta\cos^2\varphi - 1\right) \frac{R^5\left(t\right)}{5} \,\mathrm{d}\left(\cos\theta\right) \mathrm{d}\varphi \\
&= \frac{\rho R_0^5}{5} \int_0^{2\pi} \int_{-1}^{1} \left(3\left(1 - x^2\right)\cos^2\varphi - 1\right) \left(1 + \frac{a}{R_0}\sin\left(\omega t\right)\left(2x^2 - 1\right)\right)^5 \mathrm{d}x \,\mathrm{d}\varphi \\
&= \frac{\rho R_0^5}{5} \int_{-1}^{1} \left(3\pi - 3\pi x^2 - 2\pi\right) \left(1 + \frac{a}{R_0}\sin\left(\omega t\right)\left(1 - 2x^2\right)\right)^5 \mathrm{d}x \\
&= \frac{\rho\pi R_0^5}{5} \int_{-1}^{1} \left(1 - 3x^2\right) \left(1 + \frac{a}{R_0}\sin\left(\omega t\right)\left(1 - 2x^2\right)\right)^5 \mathrm{d}x.
\end{aligned}
$$

Approximating

$$
\left(1 + x\right)^n \approx 1 + nx,
$$

we obtain

$$
\begin{aligned}
Q_{xx} &\approx \frac{\rho\pi R_0^5}{5} \int_{-1}^{1} \left(1 - 3x^2\right) \left(1 + \frac{5a}{R_0}\sin\left(\omega t\right)\left(2x^2 - 1\right)\right) \mathrm{d}x \\
&= \frac{\rho\pi R_0^5}{5} \int_{-1}^{1} \left(1 - 3x^2 + \frac{5a}{R_0}\sin\left(\omega t\right)\left(-1 + 5x^2 - 6x^4\right)\right) \mathrm{d}x \\
&= \frac{\rho\pi R_0^5}{5} \left(2 - 3\frac{2}{3} - \frac{5a}{R_0}\sin\left(\omega t\right)\left(2 - 5\frac{2}{3} + 6\frac{2}{5}\right)\right) \\
&= -\frac{16}{15}\rho\pi R_0^4 a \sin\left(\omega t\right).
\end{aligned}
$$

Do to symmetry we also have $Q_{yy} = Q_{xx}$. Using the trace identity of the quadrupole tensor

$$
\mathrm{tr}\left(Q\right) = 0,
$$

we find

$$
Q_{zz} = -2Q_{xx} = -\frac{32}{15}\rho\pi R_0^4 a \sin\left(\omega t\right).
$$

The off-diagonal components vanish, for example

$$
Q_{xy} = \rho \int \left(3r'^2 \sin^2\theta \cos\varphi \sin\varphi\right) r'^2 \,\mathrm{d}r' \,\mathrm{d}\left(\cos\theta\right) \mathrm{d}\varphi = 0,
$$

since $\int_0^{2\pi} \cos\varphi \sin\varphi \,\mathrm{d}\varphi = 0$. Similarly, since $\int_0^{2\pi} \cos\varphi \,\mathrm{d}\varphi = \int_0^{2\pi} \sin\varphi \,\mathrm{d}\varphi = 0$ one finds $Q_{yz} = Q_{xz} = 0$. The angular distribution of power is then

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{1}{144\pi c^5} \left| \dddot{\vec{Q}} \times \hat{\mathbf{n}} \right|^2,$$

where we used

$$\vec{B} = \frac{\dddot{\vec{Q}} \times \hat{\mathbf{n}}}{6rc^3},$$

and

$$\left|\vec{S}\right| = \frac{c}{4\pi} |B|^2 = \frac{\mathrm{d}P}{r^2 \,\mathrm{d}\Omega},$$

and where

$$\left(\vec{Q}\right)_i = Q_{ij} n_j.$$

We have

$$\begin{aligned} Q_x &= Q_{xx} n_x + \cancelto{0}{Q_{xy}} n_y + \cancelto{0}{Q_{xz}} n_z = Q_{xx} \sin\theta \cos\varphi, \\ Q_y &= Q_{yy} n_y = Q_{yy} \sin\theta \sin\varphi, \\ Q_z &= Q_{zz} \cos\theta, \end{aligned}$$

and since $Q_{xx}, Q_{yy}, Q_{zz} \propto \sin(\omega t)$, their third derivative is

$$\dddot{Q}_{xx} \propto -\omega^3 \cos(\omega t).$$

We compute the cross product next,

$$\begin{aligned} \frac{\mathrm{d}P}{\mathrm{d}\Omega} &= \frac{\omega^6 \cos^2(\omega t)}{144\pi c^5} \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ Q_{xx,0} \sin\theta \cos\varphi & Q_{yy,0} \sin\theta \sin\varphi & Q_{zz,0} \cos\theta \\ \sin\theta \cos\varphi & \sin\theta \sin\varphi & \cos\theta \end{vmatrix}^2 \\ &= \frac{\omega^6 \cos^2(\omega t)}{144\pi c^5} \left\{ \hat{\mathbf{x}} \left(Q_{yy,0} \sin\theta \sin\varphi \cos\theta - Q_{zz,0} \cos\theta \sin\theta \sin\varphi \right) \right. \\ &- \hat{\mathbf{y}} \left(Q_{xx,0} \sin\theta \cos\varphi \cos\theta - Q_{zz,0} \cos\theta \sin\theta \cos\varphi \right) \\ &\left. + \hat{\mathbf{z}} \sin\theta \left(Q_{xx,0} \sin\theta \cos\varphi \sin\varphi - Q_{yy,0} \sin\theta \sin\varphi \cos\varphi \right) \right\}^2 \\ &= \frac{\omega^6 Q_{xx,0}^2 \cos^2(\omega t)}{144\pi c^5} 9 \sin^2\theta \cos^2\theta, \end{aligned}$$

where we used

$$Q_{xx,0} = Q_{yy,0} = -\frac{1}{2} Q_{zz,0}.$$

For the total power, we use

$$\left\langle \cos^2(\omega t)\right\rangle_T = \frac{1}{2},$$

where $T = 2\pi/\omega$, and obtain

$$\begin{aligned}
\langle P\rangle_T &= \int \mathrm{d}\Omega \left\langle \frac{\mathrm{d}P}{\mathrm{d}\Omega}\right\rangle_T \\
&= \frac{\omega^6 Q_{xx,0}^2}{2\cdot 144\pi c^5}\int \mathrm{d}\varphi\,\mathrm{d}\left(\cos\theta\right)\left[9\left(1-\cos^2\theta\right)\cos^2\theta\right] \\
&= \frac{9\omega^6 Q_{xx,0}^2}{144 c^5}\int_{-1}^{1}\mathrm{d}x\left(x^2-x^4\right) \\
&= \frac{9\omega^6 Q_{xx,0}^2}{144 c^5}\int_{-1}^{1}\mathrm{d}x\left(\frac{2}{3}-\frac{2}{5}\right) \\
&= \frac{4}{15}\frac{9\omega^6 Q_{xx,0}^2}{144 c^5} = \frac{4}{15}\frac{9\omega^6}{144 c^5}\left(\frac{16}{15}\rho\pi R_0^4 a\right)^2 .
\end{aligned}$$