Class Exercise 12 - Relativity

Relativistic notation

1. A **contra-variant 4-vector** is denoted with an upper index,

$$A^\mu = \left(A^0, A^1, A^2, A^3\right),$$

while a **co-variant 4-vector** is denoted with a lower index,

$$A_\mu = \left(A_0, A_1, A_2, A_3\right).$$

In order to sum using Einstein's summation notation, the summed terms must appear with the same upper and lower index, for example, in a coordinate transformation

$$A'^\mu = \sum_{\nu=0}^{3} \frac{\partial x'^\mu}{\partial x^\nu} A^\nu \equiv \frac{\partial x'^\mu}{\partial x^\nu} A^\nu,$$

$$A_{\prime\mu} = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu.$$

2. The (special) relativistic interval is given by

$$ds^2 = -\left(cdt\right)^2 + dx^2 + dy^2 + dz^2,$$

and can be expressed in terms of a generic displacement vector $dx^\mu = (cdt, dx, dy, dz)$ by using the **Minkowski metric tensor**

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

to **raise and lower indices**, for example

$$A_\nu = \eta_{\mu\nu} A^\mu, A^\alpha = \eta^{\alpha\beta} A_\beta.$$

The metric tensor is symmetric,

$$\eta_{\mu\nu} = \eta_{\nu\mu},$$

and has an inverse which is equal to itself (in the particular case of the Minkoski metric)

$$\eta_{\mu\nu} = \eta^{\mu\nu}.$$

The interval, as scalar, is defined as the scalar product of the displacement vector with itself, under the rules of scalar product defined by the metric,

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = dx^\mu dx_\mu.$$

Note - if an expression contains two identical indices, one of them must be an upper index and the other a lower index. There cannot be two identical upper indices, nor can there be more than two identical indices; three identical indices are not allowed!
We find

$$\begin{aligned} ds^2 &= (cdt, dx, dy, dz) \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} cdt \\ dx \\ dy \\ dz \end{pmatrix} \\ &= (cdt, dx, dy, dz) \begin{pmatrix} -cdt \\ dx \\ dy \\ dz \end{pmatrix} \\ &= -(cdt)^2 + dx^2 + dy^2 + dz^2. \end{aligned}$$

In the rest frame of some massive object $dx = dy = dz$ and thus

$$ds^2 = -c^2 d\tau^2.$$

Since the interval is a scalar, which is a relativistic invariant, this is true for any frame of reference. For any frame of reference moving at velocity v relative to the rest frame, we can write

$$ds^2 = -(cdt)^2 + dx^2 + dy^2 + dz^2 = -(cdt)^2 \left(1 - \frac{v^2}{c^2}\right) \equiv -\frac{c^2 dt^2}{\gamma^2}.$$

Again, since the interval is invariant, we find the time-dilation relation

$$d\tau = \frac{dt}{\gamma}.$$

3. The Lorentz transformation of a system moving with velocity $\beta = \upsilon/c$ in the x axis is given by operating with the matrix

$$\frac{\partial x'^{\mu}}{\partial x^{\upsilon}} = \Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

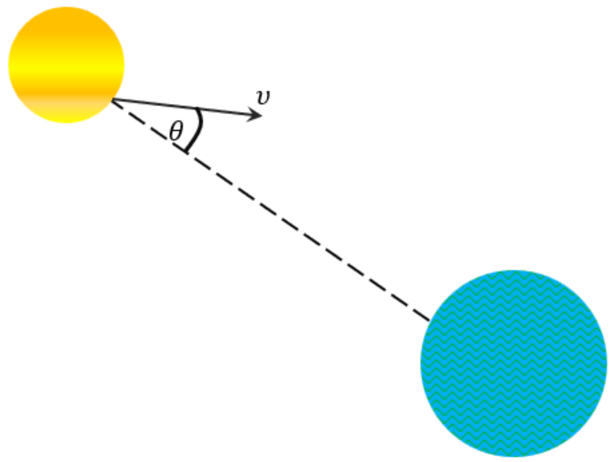
on any 4-vector we need to transform between frames of reference. The upper index μ denotes the rows and the lower index ν denotes the columns. For example, the transformed coordinates are

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma\,(ct - \beta x) \\ \gamma\,(x - \beta ct) \\ y \\ z \end{pmatrix}.$$

Problem 1

Every two years (approximately), the New York Times journal publishes an article where an astronomer claims to have found an object moving faster than the speed of light.

Consider a star that is moving at a velocity v very close to that of light c, at an angle which is small relative to the line of sight from earth (see figure). Given that the star emits light for a total duration Δt, what is the velocity v an observer on earth measures?



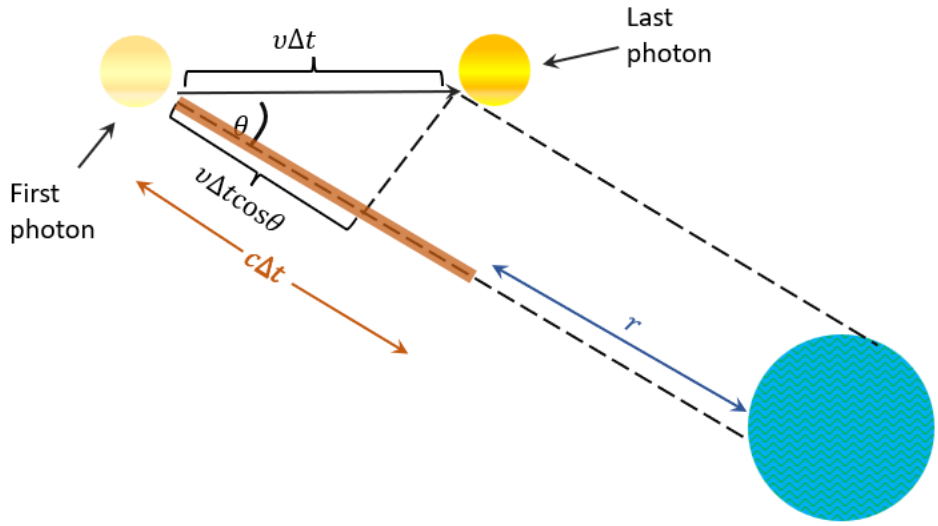
Solution

We work exclusively in the earth's frame of reference. Let us begin by finding for how long an observer on earth sees the star emit light.

The first photon from the star reaches a distance $c\Delta t$ when the last photon has been emitted from the star. Suppose that at time Δt the first photon still has a distance r to travel to earth, such that the remainder of its journey to earth will take a time $t_{\text{first}} = r/c$. The last photon will then take a time of

$$t_{\text{last}} = \frac{c\Delta t - v\Delta t\cos\theta}{c} + \frac{r}{c}$$

to reach earth.



An observer on earth will therefore measure the length of time of the emission as the difference between the first and last received photons,

$$\Delta\tau = t_{\text{last}} - t_{\text{first}} = \frac{c\Delta t - v\Delta t\cos\theta}{c} + \frac{r}{c} - \frac{r}{c} = \Delta t\left(1 - \frac{v}{c}\cos\theta\right).$$

Since the star is very far away compared to the earth, the observer sees the motion of the star as being entirely perpendicular to the line of sight and completely miss the component along the line of sight. He will therefore think that the star has traveled a distance

$$d = v\Delta t\sin\theta,$$

and "measure" an apparent velocity

$$u = \frac{d}{\Delta\tau} = \frac{v\Delta t\sin\theta}{\Delta t\left(1 - \frac{v}{c}\cos\theta\right)} = \frac{v\sin\theta}{1 - \frac{v}{c}\cos\theta}.$$

So, how can it be that an observer can deduce that the velocity exceeds the speed of light? Taking θ to be small gives

$$u \approx \frac{v\theta}{1 - \frac{v}{c} + \frac{v}{2c}\theta^2},$$

and we note that

$$1 - \frac{v}{c} = 1 - \beta = \frac{1-\beta^2}{1+\beta} = \frac{1}{\gamma^2(1+\beta)} \overset{\beta\approx 1}{\approx} \frac{1}{2\gamma^2},$$

which gives an apparent velocity of

$$u \approx \frac{v\theta}{\frac{1}{2\gamma^2} + \frac{v}{2c}\theta^2} = 2\gamma^2\frac{v\theta}{1 + \frac{\gamma^2 v}{c}\theta^2}.$$

At $\theta = c/v\gamma$, this function attains a maximum and one finds

$$u \approx \frac{2\gamma c}{1+\frac{c}{v}} \overset{v\approx c}{\approx} \gamma c \gg c!$$

Problem 2

A mirror is moving right with velocity υ. We send an electromagnetic wave with frequency ω at an angle θ relative to the mirror.

Find the frequency and direction of the reflected wave.

Solution

The mirror flips the sign of the k_x component of the wavevector vector k in its *own frame of reference*. The frequency and the angle are measured in the laboratory frame.

Let us first consider the mirror's frame of reference. It sees an initial wave with wavevector that is Lorentz transformed with respect to that of the lab,

$$k_{\text{lab}}^{\alpha} = \left(\frac{\omega}{c}, -k\cos\theta, k\sin\theta, 0\right),\ k = \frac{\omega}{c},$$

where we have constructed the wave 4-vector to give

$$k^{\mu}x_{\mu} = \vec{k}\cdot\vec{r} - \omega t,$$

such that a wave may now be written in the form

$$E^{\mu}\left(x^{\alpha}\right) = E_0^{\mu}\exp\left[ik^{\alpha}x_{\alpha}\right].$$

The scalar product is obviously invariant, which means under transformation of coordinates from the lab frame to the mirror's frame,

$$x_{\alpha} \to \Lambda_{\beta}{}^{\alpha}x_{\beta},$$

and in order to keep this scalar product invariant, or in other words, have

$$k^{\mu}x_{\mu} = k'^{\nu}x_{\prime\nu},$$

we need to transform k^{μ} as well, with the **inverse** transformation,

$$k^{\alpha} \to \Lambda^{\alpha}{}_{\beta}k^{\beta}.$$

Therefore, in the mirror's frame of reference we will see

$$k^{\alpha}_{\text{mirror}} = \Lambda^{\alpha}_{\beta} k^{\beta}_{\text{lab}} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \frac{\omega}{c} \begin{pmatrix} 1 \\ -\cos\theta \\ \sin\theta \\ 0 \end{pmatrix} = \frac{\omega\gamma}{c} \begin{pmatrix} 1+\beta\cos\theta \\ -\beta-\cos\theta \\ \gamma^{-1}\sin\theta \\ 0 \end{pmatrix}.$$

We can only flip the sign of the x component of the wavevector in the mirror's frame. We find

$$(k^{\alpha}_{\text{mirror}})^{(\text{reflected})} = \frac{\omega\gamma}{c} \begin{pmatrix} 1+\beta\cos\theta \\ \beta+\cos\theta \\ \gamma^{-1}\sin\theta \\ 0 \end{pmatrix}.$$

Now we can transform back to obtain the reflected wave in the lab's frame,

$$\left(k^{\beta}_{\text{lab}}\right)^{(\text{reflected})} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \frac{\omega\gamma}{c} \begin{pmatrix} 1+\beta\cos\theta \\ \beta+\cos\theta \\ \gamma^{-1}\sin\theta \\ 0 \end{pmatrix} = \frac{\omega\gamma^2}{c} \begin{pmatrix} 1+\beta\cos\theta+\beta\left(\beta+\cos\theta\right) \\ \beta\left(1+\beta\cos\theta\right)+\beta+\cos\theta \\ \gamma^{-2}\sin\theta \\ 0 \end{pmatrix},$$

which is equal to

$$\left(k^{\beta}_{\text{lab}}\right)^{(\text{reflected})} = \left(\frac{\omega_{\text{ref}}}{c}, \left(k_x\right)_{\text{ref}}, \left(k_y\right)_{\text{ref}}, 0\right).$$

Equating the result with this form, we find that the frequency of the wave has been blue shifted,

$$\omega_{\text{ref}} = \omega\gamma^2\left(1+2\beta\cos\theta+\beta^2\right) > \omega.$$

The direction of the reflected wave has also changed, and is now

$$\tan\theta_{\text{ref}} = \frac{\left(k_y\right)_{\text{ref}}}{\left(k_x\right)_{\text{ref}}} = \frac{\gamma^{-2}\sin\theta}{\beta\underbrace{\left(2+\beta\cos\theta\right)}_{>0}+\cos\theta} < \frac{\gamma^{-2}\sin\theta}{\cos\theta} < \tan\theta.$$

Problem 3

1. We introduce the Field tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}.$$

We define the potential 4-vector

$$A^\mu = (\phi, A_x, A_y, A_z).$$

Remark - the Lorentz gauge in this notation is given by the scalar equation

$$\partial_\mu A^\mu = \frac{1}{c}\frac{\partial \phi}{\partial t} + \vec{\nabla} \cdot \vec{A} = 0.$$

Show that the field tensor can be written as the anti-symmetric derivative of the potential 4-vector,

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu.$$

2. The Maxwell equations in relativistic notation can be shortened from 4 equations to only 2,

$$\partial_\nu F^{\mu\nu} = \frac{4\pi}{c} J^\mu,$$
$$\partial_\nu \left(\varepsilon^{\alpha\beta\nu\mu} F_{\alpha\beta}\right) = 0,$$

under the definition

$$J^\mu = \left(\rho c, \vec{j}\right),$$

and a definition of $\varepsilon^{\alpha\beta\nu\mu}$ that is completely equivalent to the 3d case. Show that the first equation corresponds to the pair of non-homogeneous equations

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho, \quad \vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} - \frac{4\pi}{c}\vec{j},$$

while the second corresponds to

$$\vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0.$$

Solution

1. We check the F^{0i} components,

$$\begin{aligned} F^{0i} &= \partial^0 A^i - \partial^i A^0 = \eta^{00}\partial_0 A^i - \eta^{ii}\partial_i A^0 \\ &= (-1)\,\frac{\partial A^i}{\partial x^0} - (+1)\,\frac{\partial A^0}{\partial x^i} \\ &= -\frac{1}{c}\frac{\partial A^i}{\partial t} - \frac{\partial \phi}{\partial x^i} \\ &= E_i. \end{aligned}$$

   We now check the F^{12},

$$\begin{aligned} F^{12} &= \partial^1 A^2 - \partial^2 A^1 = \eta^{11}\partial_1 A^2 - \eta^{22}\partial_2 A^1 \\ &= \frac{\partial A^y}{\partial x} - \frac{\partial A^x}{\partial y} \\ &= \left(\vec{\nabla} \times \vec{A}\right)_z \\ &= B_z. \end{aligned}$$

   The same is easily checked for any of the F^{ij} components. The components on the diagonal obviously vanish due to the anti-symmetric form of $F^{\mu\nu}$.

2. We check the first of the two equations,

$$\partial_\nu F^{\mu\nu} = \frac{4\pi}{c} J^\mu.$$

   Note that $\mu = 0, 1, 2, 3$ and this is infact a set of 4 equations. For $\mu = 0$ we find

$$\partial_\nu F^{0\nu} = \partial_\nu \delta_i^\nu E^i = \frac{4\pi}{c} J^0 = \frac{4\pi}{c} c\rho,$$

   and indeed obtain the first equation

$$\partial_\nu E^\nu = \vec{\nabla} \cdot \vec{E} = 4\pi\rho.$$

Now, for $\mu = 1$ for example, we have on the LHS

$$\begin{aligned}
\partial_\nu F^{1\nu} &= \frac{1}{c}\partial_t F^{10} + \partial_1 F^{11} + \partial_2 F^{12} + \partial_3 F^{13} \\
&= \frac{1}{c}\frac{\partial E_x}{\partial t} + \frac{\partial B_z}{\partial y} + \frac{\partial\left(-B_y\right)}{\partial z} \\
&= \left(\frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \vec{\nabla}\times\vec{B}\right)_x,
\end{aligned}$$

and on the RHS we have

$$\frac{4\pi}{c}J^1 = \frac{4\pi}{c}j_x,$$

so essentially we have found the x component of a the second vector equation. A similar computation for $\mu = 2, 3$ yields the rest of the components of this equation. Now for the second of the relativistic Maxwell equations,

$$\partial_\nu\left(\varepsilon^{\alpha\beta\nu\mu}F_{\alpha\beta}\right) = 0.$$

Direct computations lead to

$$G^{\mu\nu} = \frac{1}{2}\varepsilon^{\alpha\beta\nu\mu}F_{\alpha\beta} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{pmatrix},$$

and we use this form to compute the components of the equation. Taking $\mu = 0$ first, we find the first homogeneous Maxwell equation,

$$\partial_\nu G^{0\nu} = \partial_x B_x + \partial_y B_y + \partial_z B_z = \vec{\nabla}\cdot\vec{B} = 0.$$

Taking $\mu = 1$ for example, we find a component of the second homogeneous equation,

$$\begin{aligned}
\partial_\nu G^{1\nu} &= \frac{1}{c}\frac{\partial B_x}{\partial t} - \frac{\partial E_z}{\partial y} + \frac{\partial E_y}{\partial z} \\
&= \left(\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t}\right)_x,
\end{aligned}$$

and the rest of the components are similarly gotten for $\mu = 2, 3$.