Class Exercise 13 - Relativity

Electrodynamics in relativistic notation

Important 4-vectors:

- Coordinate location 4-vector

$$x^\mu = (ct, x, y, z) = (ct, \vec{x}),$$

- Velocity 4-vector

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma\left(c\frac{\mathrm{d}t}{\mathrm{d}t}, \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t}\right) = \gamma\left(c, \upsilon_x, \upsilon_y, \upsilon_z\right) = \gamma\left(c, \vec{\upsilon}\right),$$

- Momentum 4-vector

$$p^\mu = mu^\mu = m\gamma\left(c, \vec{\upsilon}\right) = \left(\frac{E}{c}, \vec{p}\right).$$

The 4-velocity is **always** normalized:

$$g_{\mu\nu}u^\mu u^\nu = u^\mu u_\mu = \gamma\left(c, \vec{\upsilon}\right) \cdot \gamma\left(-c, \vec{\upsilon}\right) = \gamma^2\left(-c^2 + \upsilon^2\right) = \frac{-c^2}{1 - \frac{\upsilon^2}{c^2}}\left(1 - \frac{\upsilon^2}{c^2}\right) = -c^2.$$

Therefore, $u^\mu u_\mu = -c^2$ and as a consequence,

$$p_\mu p^\mu = m^2 u^\mu u_\mu = -m^2c^2,$$

and since on the other hand,

$$p_\mu p^\mu = -\left(\frac{E}{c}\right)^2 + p^2,$$

we get the invariant energy scalar,

$$E^2 - p^2c^2 = m^2c^4.$$

- Derivative 4-vector,

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right),$$

- Potential 4-vector

$$A^\mu = \left(\phi, \vec{A}\right),$$

and current 4-vector,

$$J^\mu = \left(\rho c, \vec{j}\right),$$

which satisfies the continuity equation

$$\partial_\mu J^\mu = \frac{1}{c}\frac{\partial\left(\rho c\right)}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0.$$

Together, these allow us to write the Maxwell equations in the form

$$\partial_\nu F^{\mu\nu} = \frac{4\pi}{c} J^\mu,$$
$$\partial_\nu \left(\varepsilon^{\alpha\beta\nu\mu} F_{\alpha\beta}\right) = 0,$$

where

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$$

is the field tensor, defined as

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}.$$

Problem 1

A physicist uses a particle accelerator to collide a pair of particles with masses m_1, m_2. The physicist wants to do an experiment with as much center of mass energy E_{CM} as possible. The highest energy with which he can accelerate the particles *in the lab frame* is

$$2E_0 \gg m_1 c^2, m_2 c^2.$$

How should the physicist perform the experiment?

1. Collide one particle with energy $2E_0$ with a second particle at rest,
2. Collide both particles at equal energies E_0 and opposite directions.

Solution

Note that

$$\left(p^\mu_{\text{Total}}\right)^2 \equiv s = (p_1^\mu + p_2^\mu)^2$$

is an invariant, a constant in the center of mass frame and in the lab frame. In the center of mass frame, we have

$$(p_1^\mu)_{CM} = \left(\frac{(E_1)_{CM}}{c}, (\vec{p}_1)_{CM}\right),$$
$$(p_2^\mu)_{CM} = \left(\frac{(E_2)_{CM}}{c}, (\vec{p}_2)_{CM}\right),$$

and thus

$$(p_1^\mu + p_2^\mu)_{CM} = \left(\frac{(E_1)_{CM} + (E_2)_{CM}}{c}, (\vec{p}_1)_{CM} + (\vec{p}_2)_{CM}\right).$$

But, in the center of mass frame,

$$(\vec{p}_1)_{CM} + (\vec{p}_2)_{CM} = 0,$$

and thus

$$(p_1^\mu + p_2^\mu)_{CM} = \frac{(E_1)_{CM} + (E_2)_{CM}}{c} \equiv \frac{E_{CM}}{c}.$$

Therefore,

$$s_{CM} = -\frac{E_{CM}^2}{c^2}.$$

Let us consider each scenario separately, to understand which of them has the highest center of mass energy.

1. The second particle is at rest, so we have the 4-momenta

$$(p_1^\mu)_{\text{lab}} = \left(\frac{2E_0}{c}, p_0, 0, 0\right),$$
$$(p_2^\mu)_{\text{lab}} = \left(\frac{m_2c^2}{c}, 0, 0, 0\right),$$

where

$$(2E_0)^2 = p_0^2c^2 + m_1^2c^4.$$

Therefore,

$$\begin{aligned} s_{\text{lab}} = (p_1^\mu + p_2^\mu)^2_{\text{lab}} &= \left(\frac{2E_0}{c} + m_2c, p_0, 0, 0\right)^2 \\ &= -\left(\frac{2E_0}{c} + m_2c\right)^2 + p_0^2 \\ &= -\frac{4E_0^2}{c^2} - \frac{4E_0}{c}m_2c - m_2^2c^2 + \underbrace{\frac{4E_0^2}{c^2} - m_1^2c^2}_{=p_0^2} \\ &= -4E_0m_2 - \left(m_2^2 + m_1^2\right)c^2. \end{aligned}$$

Comparing $s_{\text{lab}} = s_{CM}$ we find

$$E_{CM} = \sqrt{4E_0m_2c^2 + \left(m_2^2 + m_1^2\right)c^4}.$$

In the (ultra-relativistic) limit $E_0 \gg m_1c^2, m_2c^2$ we have the center of mass energy

$$E_{CM} \approx 2\sqrt{E_0m_2c^2}.$$

2. In this case, the lab and center of mass frames are one and the same, since in both of them

$$\vec{p_1} + \vec{p_2} = 0.$$

Therefore, we have the center of mass energy

$$E_{CM} = 2E_0,$$

which is much larger than the one obtained in method (1), since in the ultra-relativistic limit of $E_0 \gg m_1c^2, m_2c^2$ we have

$$\frac{2E_0}{2\sqrt{E_0m_2c^2}} = \sqrt{\frac{E_0}{m_2c^2}} \gg 1.$$

Let us consider the actual numbers for a moment: The highest energy in the LHC is

$$E_0 = 6,500\,[GeV]\,.$$

For a proton beam with $m_p c^2 \approx 0.938\,[GeV]$, we have an energy ratio of

$$\sqrt{\frac{E_0}{m_2 c^2}} \approx 83.2,$$

which means, a head-on collision of two proton beams has 83 times the center of mass energy than a collision of a moving particle with a particle at rest. That is precisely the reason that the experiments in the LHC are done in method (2).

Problem 2

Show that $\vec{E}\cdot\vec{B}$ is invariant under Lorentz transformations (in the particular case of a system moving in $+\upsilon\hat{\mathbf{x}}$).

Solution

We use the fields and Lorentz transformation tensors

$$F^{\mu\nu}=\begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix},\quad \frac{\partial x'^{\mu}}{\partial x^{\upsilon}}=\Lambda^{\mu}{}_{\nu}=\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

to prove that $\vec{E}\cdot\vec{B}=E_xB_x+E_yB_y+E_zB_z$ is the same as $\vec{E}'\cdot\vec{B}'$:

$$\begin{aligned} E'_xB'_x &= F'^{01}F'^{23}=\left(\Lambda^{0}{}_{\alpha}\Lambda^{1}{}_{\beta}F^{\alpha\beta}\right)\left(\Lambda^{2}{}_{\gamma}\Lambda^{3}{}_{\delta}F^{\gamma\delta}\right) \\ &= \left(\Lambda^{0}{}_{0}\Lambda^{1}{}_{1}F^{01}+\Lambda^{0}{}_{1}\Lambda^{1}{}_{0}F^{10}\right)\left(\Lambda^{2}{}_{2}\Lambda^{3}{}_{3}F^{23}\right) \\ &= \left(\gamma\gamma E_x+(-\gamma\beta)(-\gamma\beta)(-E_x)\right)B_x \\ &= \gamma^2\left(1-\beta^2\right)E_xB_x=E_xB_x, \end{aligned}$$

$$\begin{aligned} E'_yB'_y &= F'^{02}F'^{31}=\left(\Lambda^{0}{}_{\alpha}\Lambda^{2}{}_{\beta}F^{\alpha\beta}\right)\left(\Lambda^{3}{}_{\gamma}\Lambda^{1}{}_{\delta}F^{\gamma\delta}\right) \\ &= \left(\Lambda^{0}{}_{0}\Lambda^{2}{}_{2}F^{02}+\Lambda^{0}{}_{1}\Lambda^{2}{}_{2}F^{12}\right)\left(\Lambda^{3}{}_{3}\Lambda^{1}{}_{0}F^{30}+\Lambda^{3}{}_{3}\Lambda^{1}{}_{1}F^{31}\right) \\ &= \left(\gamma E_y-\gamma\beta B_z\right)\left((-\gamma\beta)(-E_z)+\gamma B_y\right) \\ &= \gamma^2\left(E_y-\beta B_z\right)\left(\beta E_z+B_y\right) \\ &= \gamma^2\left(\beta E_yE_z+E_yB_y-\beta^2B_zE_z-\beta B_zB_y\right), \end{aligned}$$

$$\begin{aligned} E'_zB'_z &= F'^{03}F'^{12}=\left(\Lambda^{0}{}_{\alpha}\Lambda^{3}{}_{\beta}F^{\alpha\beta}\right)\left(\Lambda^{1}{}_{\gamma}\Lambda^{2}{}_{\delta}F^{\gamma\delta}\right) \\ &= \left(\Lambda^{0}{}_{0}\Lambda^{3}{}_{3}F^{03}+\Lambda^{0}{}_{1}\Lambda^{3}{}_{3}F^{13}\right)\left(\Lambda^{1}{}_{0}\Lambda^{2}{}_{2}F^{02}+\Lambda^{1}{}_{1}\Lambda^{2}{}_{2}F^{12}\right) \\ &= \left(\gamma E_z+(-\gamma\beta)(-B_y)\right)\left(-\gamma\beta E_y+\gamma B_z\right) \\ &= \gamma^2\left(-\beta E_zE_y+E_zB_z-\beta^2B_yE_y+\beta B_yB_z\right). \end{aligned}$$

Summing up over all these terms yields

$$\begin{aligned}\vec{E}' \cdot \vec{B}' &= E_x B_x + \gamma^2 \left(\cancel{\beta E_y E_z} + E_y B_y - \beta^2 B_z E_z - \cancel{\beta B_z B_y} \right) + \gamma^2 \left(-\cancel{\beta E_z E_y} + E_z B_z - \beta^2 B_y E_y + \cancel{\beta B_y B_z} \right) \\ &= E_x B_x + \gamma^2 \left(\left(1 - \beta^2\right) E_y B_y + \left(1 - \beta^2\right) B_z E_z \right) \\ &= E_x B_x + E_y B_y + B_z E_z = \vec{E} \cdot \vec{B}.\end{aligned}$$

Indeed, this expression is Lorentz invariant!

Problem 3

Consider a static wire charged with uniform line charge density λ. Find the electric and magnetic fields an observer in a frame of reference that is boosted at velocity v (along the axis of the wire) measures.

Solution

As we well know by now, the electric field of a charged wire (which we take to lie along the z axis) is

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 r}\hat{\mathbf{r}} = \frac{\lambda}{2\pi\epsilon_0\left(x^2+y^2\right)}\begin{pmatrix} x \\ y \\ 0 \end{pmatrix},$$

while the magnetic field vanishes in the static frame, $\vec{B}=0$. Therefore, in the lab frame of reference we have a field tensor

$$F^{\mu\nu} = \frac{1}{c}\begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & 0 & 0 \\ -E_y & 0 & 0 & 0 \\ -E_z & 0 & 0 & 0 \end{pmatrix}.$$

Let us transform to the moving frame, where

$$F'^{\alpha\beta} = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu F^{\mu\nu}.$$

The diagonal elements are vanishing,

$$\begin{aligned} F'^{\alpha\alpha} = \Lambda^\alpha{}_\mu \Lambda^\alpha{}_\nu F^{\mu\nu} &= \Lambda^0{}_0\Lambda^0{}_3 F^{03} + \Lambda^0{}_3\Lambda^0{}_0 F^{30} + \Lambda^3{}_0\Lambda^3{}_3 F^{03} + \Lambda^3{}_3\Lambda^3{}_0 F^{30} \\ &\quad + \Lambda^1{}_1\Lambda^1{}_1 \underbrace{\delta^\mu_1\delta^\nu_1 F^{\mu\nu}}_{=F^{11}=0} + \Lambda^2{}_2\Lambda^2{}_2 \underbrace{\delta^\mu_2\delta^\nu_2 F^{\mu\nu}}_{=F^{22}=0} \\ &= \Lambda^0{}_0\Lambda^0{}_3\left(F^{03}-F^{03}\right) + \Lambda^3{}_0\Lambda^3{}_3\left(F^{03}-F^{03}\right) = 0. \end{aligned}$$

Focusing on the magnetic field, we calculate its components $B_x = F^{23}, B_y = F^{31}, B_z = F^{12}$:

$$\begin{aligned} B'_x = F'^{23} = \Lambda^2{}_\mu \Lambda^3{}_\nu F^{\mu\nu} &= \Lambda^2{}_2\Lambda^3{}_0 F^{20} + \Lambda^2{}_2\Lambda^3{}_3 F^{23} \\ &= \left(-\gamma\beta\right)\left(-E_y/c\right) \\ &= \frac{\gamma\beta}{c}E_y, \end{aligned}$$

$$\begin{aligned}
B'_y = F'^{31} = \Lambda^3{}_\mu \Lambda^1{}_\nu F^{\mu\nu} &= \Lambda^3{}_0 \Lambda^1{}_1 F^{01} + \Lambda^3{}_3 \Lambda^1{}_1 F^{31} \\
&= (-\gamma\beta)\left(\frac{E_x}{c}\right) \\
&= -\frac{\gamma\beta}{c} E_x,
\end{aligned}$$

$$\begin{aligned}
B'_z = F'^{12} &= \Lambda^1{}_\mu \Lambda^2{}_\nu F^{\mu\nu} \\
&= \Lambda^1{}_1 \Lambda^2{}_2 F^{12} \\
&= 0.
\end{aligned}$$

As expected, we now find a non-vanishing magnetic field

$$\vec{B} = \frac{\gamma\beta\lambda}{2\pi c\epsilon_0 \left(x'^2 + y'^2\right)} \begin{pmatrix} y' \\ -x' \\ 0 \end{pmatrix} = \frac{\gamma\beta\lambda}{2\pi c\epsilon_0 \left(x^2 + y^2\right)} \begin{pmatrix} y \\ -x \\ 0 \end{pmatrix},$$

where we used the fact that $x' = x$ and $y' = y$ - these are directions perpendicular to the boost. From the perspective of an observer in the boosted frame, the charge density in the wire is

$$\lambda' = \gamma\lambda,$$

where the factor of γ is the result of a Lorentz contraction in the length of the wire. Since the charge density appears to be moving in this frame, the moving observer sees a current

$$I' = -v\lambda' = -\gamma v\lambda,$$

and an appropriate magnetic field of a wire with current,

$$\vec{B} = -\frac{\mu_0 I'}{2\pi r} \begin{pmatrix} y \\ -x \\ 0 \end{pmatrix} = -\frac{\mu_0 I'}{2\pi r} \begin{pmatrix} \sin\varphi \\ -\cos\varphi \\ 0 \end{pmatrix} = \frac{\mu_0 I'}{2\pi r} \hat{\boldsymbol{\varphi}}.$$

Both the magnetic and electric field are obtained at once using standard matrix multiplication,

$$
\begin{aligned}
F'^{\alpha\beta} = \Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}{}_{\nu}F^{\mu\nu} &= \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \frac{1}{c} \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & 0 & 0 \\ -E_y & 0 & 0 & 0 \\ -E_z & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \\
&= \frac{1}{c} \begin{pmatrix} \left(-\gamma\beta\right)\left(-E_z\right) & \gamma E_x & \gamma E_y & \gamma E_z \\ -E_x & 0 & 0 & 0 \\ -E_y & 0 & 0 & 0 \\ -\gamma E_z & -\gamma\beta E_x & -\gamma\beta E_y & -\gamma\beta E_z \end{pmatrix} \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \\
&= \frac{1}{c} \begin{pmatrix} \left(-\gamma\beta\right)\left(-E_z\right)\gamma + \left(-\gamma\beta\right)\gamma E_z & \gamma E_x & \gamma E_y & \left(-\gamma\beta\right)^2\left(-E_z\right) + \gamma^2 E_z \\ -\gamma E_x & 0 & 0 & \left(-E_x\right)\left(-\gamma\beta\right) \\ -\gamma E_y & 0 & 0 & \left(-E_y\right)\left(-\gamma\beta\right) \\ -\gamma^2 E_z + \left(-\gamma\beta\right)^2 E_z & -\gamma\beta E_x & -\gamma\beta E_y & \left(-\gamma E_z\right)\left(-\gamma\beta\right) - \gamma^2\beta E_z \end{pmatrix} \\
&= \frac{1}{c} \begin{pmatrix} 0 & \gamma E_x & \gamma E_y & E_z \\ -\gamma E_x & 0 & 0 & \gamma\beta E_x \\ -\gamma E_y & 0 & 0 & \gamma\beta E_y \\ -E_z & -\gamma\beta E_x & -\gamma\beta E_y & 0 \end{pmatrix}.
\end{aligned}
$$

Comparing this form with the standard one for $F^{\mu\nu}$, we find

$$
\begin{aligned}
&E'_x = \gamma E_x,\ E'_y = \gamma E_y,\ E'_z = E_z = 0, \\
&B'_x = \frac{\gamma\beta}{c} E_y,\ B'_y = -\frac{\gamma\beta}{c} E_x,\ B'_z = B_z = 0.
\end{aligned}
$$