Homework 12

Question 1

1. Prove that a symmetric tensor remains symmetric under Lorentz transformations. Prove the same for an antisymmetric tensor.

2. Verify that the Minkowski metric is invariant under Lorentz transformations,

$$\eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \eta'^{\mu\nu},$$

   for the case of a system moving with velocity $\vec{v} = v_x \hat{\mathbf{x}}$. Convince yourselves that, since the x axis was chosen to coincide with the direction of the velocity, this is a generic property of $\eta^{\mu\nu}$.

Question 2

Consider an electromagnetic field in the lab frame of reference,

$$\vec{E} = E_0 \hat{\mathbf{z}}, \quad \vec{B} = B_0 \hat{\mathbf{x}},$$

where $|B_0| > |E_0|$.

1. Show that $-\frac{1}{2} F^{\mu\nu} F_{\mu\nu} = E^2 - B^2$ and explain why this quantity is the same in all coordinate systems. **Hint**: You may find it useful to use the definitionTr $(M) \equiv M^{\mu\nu} g_{\mu\nu}$ in your calculations.

2. Find the velocity $\vec{v} = v\hat{\mathbf{y}}$ of a system S' in which the electric field vanishes. Find the magnetic field that is measured in this frame of reference.

3. Would there still be a fame of reference where $E' = 0$ if $|E_0| > |B_0|$?

Question 3

1. Find the coordinate transformation matrix $\Lambda^{\mu}{}_{\nu}$ for a reflection transformation on the z axis,
$$(ct, x, y, z) \mapsto (ct, x, y, -z)\,.$$

2. Show how the electromagnetic field tensor $F^{\mu\nu}$ transforms under this transformation. Do the fields $\vec{E}, \vec{B}$ transform like $3-$vectors (in other words, like a vector in 3 spatial dimensions would transform with respect to $\Lambda^{i}{}_{j}$ where $i, j = 1, 2, 3$)?

Question 4

Consider a point charge q moving at a constant velocity $\vec{v} = v\hat{\mathbf{z}}$, which is located at the origin at time $t = 0$ in the lab frame S.

Use a Lorentz transformation from the particle's rest frame S' to the lab frame of reference S, and find the electromagnetic fields as measured in the lab.

Question 5

A parallel-plate capacitor, at rest in the frame S and tilted at a 45°angle with respect to the x axis, carries charge densities $\pm\sigma_0$ on the two plates. System S' is moving to the right on the x axis, at speed v, relative to S.

1. Find E, the electric field in frame S.

2. Find E', the electric field in S'.

3. What angle do the plates make with the x axis in S'? Is the field perpendicular to the plates?