

## Introduction to Partial Differentiation

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The aim of this document is to provide a short, self assessment programme for students who wish to acquire a basic understanding of partial differentiation.

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The full range of these packages and some instructions, should they be required, can be obtained from our web page [Mathematics Support Materials](#).

## 1. Partial Differentiation (Introduction)

In the package on **introductory differentiation**, rates of change of functions were shown to be measured by the *derivative*. Many applications require functions with more than one variable: the ideal gas law, for example, is

$$pV = kT$$

where  $p$  is the pressure,  $V$  the volume,  $T$  the absolute temperature of the gas, and  $k$  is a constant. Rearranging this equation as

$$p = \frac{kT}{V}$$

shows that  $p$  is a function of  $T$  and  $V$ . If one of the variables, say  $T$ , is kept fixed and  $V$  changes, then the derivative of  $p$  with respect to  $V$  measures the *rate of change* of *pressure* with respect to *volume*. In this case, it is called *the partial derivative of  $p$  with respect to  $V$*  and written as

$$\frac{\partial p}{\partial V}.$$

**Example 1** If  $p = \frac{kT}{V}$ , find the partial derivatives of  $p$ :

- (a) with respect to  $T$ ,      (b) with respect to  $V$ .

**Solution**

(a) This part of the example proceeds as follows:

$$\begin{aligned} p &= \frac{kT}{V}, \\ \therefore \frac{\partial p}{\partial T} &= \frac{k}{V}, \end{aligned}$$

where  $V$  is treated as a constant for this calculation.

(b) For this part,  $T$  is treated as a constant. Thus

$$\begin{aligned} p &= kT \frac{1}{V} = kTV^{-1}, \\ \therefore \frac{\partial p}{\partial V} &= -kTV^{-2} = -\frac{kT}{V^2}. \end{aligned}$$

The symbol  $\partial$  is used whenever a function with more than one variable is being differentiated but the techniques of *partial* differentiation are exactly the same as for (*ordinary*) differentiation.

**Example 2** Find  $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$  for the function  $z = x^2y^3$ .

**Solution**

$$\begin{aligned} z &= x^2y^3 && \text{For the first part } y^3 \text{ is treated as} \\ &&& \text{a constant and the derivative of} \\ \therefore \frac{\partial z}{\partial x} &= 2xy^3, && \text{ } x^2 \text{ with respect to } x \text{ is } 2x. \\ \text{and } \frac{\partial z}{\partial y} &= x^23y^2, && \text{For the second part } x^2 \text{ is treated} \\ &&& \text{as a constant and the derivative} \\ &= 3x^2y^2. && \text{of } y^3 \text{ with respect to } y \text{ is } 3y^2. \end{aligned}$$

**EXERCISE 1.** Find  $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$  for each of the following functions.

(Click on the **green** letters for solutions.)

(a)  $z = x^2y^4$ ,      (b)  $z = (x^4 + x^2)y^3$ ,      (c)  $z = y^{\frac{1}{2}} \sin(x)$ .

## 2. The Rules of Partial Differentiation

Since *partial differentiation* is essentially the same as *ordinary differentiation*, the *product*, *quotient* and *chain* rules may be applied.

**Example 3** Find  $\frac{\partial z}{\partial x}$  for each of the following functions.

$$(a) z = xy \cos(xy), \quad (b) z = \frac{x-y}{x+y}, \quad (c) z = (3x+y)^2.$$

### Solution

(a) Here  $z = uv$ , where  $u = xy$  and  $v = \cos(xy)$  so the *product rule* applies (see the package on the **Product and Quotient Rules**).

$$\begin{aligned} u &= xy & \text{and} & & v &= \cos(xy) \\ \therefore \frac{\partial u}{\partial x} &= y & \text{and} & & \frac{\partial v}{\partial x} &= -y \sin(xy). \end{aligned}$$

Thus

$$\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x} v + u \frac{\partial v}{\partial x} = y \cos(xy) - xy^2 \sin(xy).$$

(b) Here  $z = u/v$ , where  $u = x - y$  and  $v = x + y$  so the *quotient rule* applies (see the package on the **Product and Quotient Rules**).

$$\begin{aligned}u &= x - y & \text{and} & & v &= x + y \\ \therefore \frac{\partial u}{\partial x} &= 1 & \text{and} & & \frac{\partial v}{\partial x} &= 1.\end{aligned}$$

Thus

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x}}{v^2} \\ &= \frac{(x + y) - (x - y)}{(x + y)^2} = \frac{2y}{(x + y)^2}.\end{aligned}$$

(c) In this case  $z = (3x + y)^2$  so  $z = u^2$  where  $u = 3x + y$ , and the *chain rule* applies (see the package on the **Chain Rule**).

$$\begin{aligned}z &= u^2 & \text{and} & & u &= 3x + y \\ \therefore \frac{\partial z}{\partial u} &= 2u & \text{and} & & \frac{\partial u}{\partial x} &= 3.\end{aligned}$$

Thus

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} = 2(3x + y)3 = 6(3x + y).$$

**EXERCISE 2.** Find  $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$  for each of the following functions.

(Click on the **green** letters for solutions.)

(a)  $z = (x^2 + 3x) \sin(y)$ ,      (b)  $z = \frac{\cos(x)}{y^5}$ ,      (c)  $z = \ln(xy)$ ,

(d)  $z = \sin(x) \cos(xy)$ ,      (e)  $z = e^{(x^2 + y^2)}$ ,      (f)  $z = \sin(x^2 + y)$ .

**Quiz** If  $z = \cos(xy)$ , which of the following statements is true?

(a)  $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$ ,

(b)  $\frac{\partial z}{\partial x} = \frac{1}{x} \frac{\partial z}{\partial y}$ ,

(c)  $\frac{1}{y} \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$ ,

(d)  $\frac{1}{y} \frac{\partial z}{\partial x} = \frac{1}{x} \frac{\partial z}{\partial y}$ .



### 3. Higher Order Partial Derivatives

Derivatives of order two and higher were introduced in the package on **Maxima and Minima**. Finding higher order derivatives of functions of more than one variable is similar to ordinary differentiation.

**Example 4** Find  $\frac{\partial^2 z}{\partial x^2}$  if  $z = e^{(x^3+y^2)}$ .

**Solution** First differentiate  $z$  with respect to  $x$ , keeping  $y$  constant, then differentiate this function with respect to  $x$ , again keeping  $y$  constant.

$$\begin{aligned}z &= e^{(x^3+y^2)} \\ \therefore \frac{\partial z}{\partial x} &= 3x^2 e^{(x^3+y^2)} \text{ using the chain rule} \\ \frac{\partial^2 z}{\partial x^2} &= \frac{\partial(3x^2)}{\partial x} e^{(x^3+y^2)} + 3x^2 \frac{\partial(e^{(x^3+y^2)})}{\partial x} \text{ using the product rule} \\ \frac{\partial^2 z}{\partial x^2} &= 6xe^{(x^3+y^2)} + 3x^2(3x^2 e^{(x^3+y^2)}) \\ &= (9x^4 + 6x)e^{(x^3+y^2)}\end{aligned}$$

In addition to both  $\frac{\partial^2 z}{\partial x^2}$  and  $\frac{\partial^2 z}{\partial y^2}$ , when there are two variables there is also the possibility of a *mixed second order derivative*.

**Example 5** Find  $\frac{\partial^2 z}{\partial x \partial y}$  if  $z = e^{(x^3+y^2)}$ .

**Solution** The symbol  $\frac{\partial^2 z}{\partial x \partial y}$  is interpreted as  $\frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right)$ ; in words, first differentiate  $z$  with respect to  $y$ , keeping  $x$  constant, then differentiate this function with respect to  $x$ , keeping  $y$  constant. (It is this differentiation, first with respect to  $x$  and then with respect to  $y$ , that leads to the name of *mixed derivative*.)

$$\text{First with } x \text{ constant} \quad \frac{\partial z}{\partial y} = 2ye^{(x^3+y^2)} \quad (\text{using the chain rule.})$$

$$\begin{aligned} \text{Second with } y \text{ constant} \quad \frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x} \left( 2ye^{(x^3+y^2)} \right) \\ &= 3x^2 2ye^{(x^3+y^2)} \\ &= 6x^2 ye^{(x^3+y^2)}. \end{aligned}$$

The obvious question now to be answered is: what happens if the order of differentiation is reversed?

**Example 6** Find  $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right)$  if  $z = e^{(x^3+y^2)}$ .

**Solution**

First with  $y$  constant  $\frac{\partial z}{\partial x} = 3x^2 e^{(x^3+y^2)}$  (using the chain rule).

Second with  $x$  constant  $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left( 3x^2 e^{(x^3+y^2)} \right)$   
 $= 2y 3x^2 e^{(x^3+y^2)}$   
 $= 6x^2 y e^{(x^3+y^2)} = \frac{\partial^2 z}{\partial x \partial y}.$

As a general rule, when calculating *mixed derivatives* the order of differentiation may be reversed without affecting the final result.

**EXERCISE 3.** Confirm the statement on the previous page by finding both  $\frac{\partial^2 z}{\partial x \partial y}$  and  $\frac{\partial^2 z}{\partial y \partial x}$  for each of the following functions, whose first order partial derivatives have already been found in [exercise 2](#). (Click on the [green](#) letters for solutions.)

- (a)  $z = (x^2 + 3x) \sin(y)$ ,      (b)  $z = \frac{\cos(x)}{y^5}$ ,      (c)  $z = \ln(xy)$ ,  
(d)  $z = \sin(x) \cos(xy)$ ,      (e)  $z = e^{(x^2 + y^2)}$ ,      (f)  $z = \sin(x^2 + y)$ .

**Notation** For first and second order partial derivatives there is a compact notation. Thus  $\frac{\partial f}{\partial x}$  can be written as  $f_x$  and  $\frac{\partial f}{\partial y}$  as  $f_y$ .

Similarly  $\frac{\partial^2 f}{\partial x^2}$  is written  $f_{xx}$  while  $\frac{\partial^2 f}{\partial x \partial y}$  is written  $f_{xy}$ .

**Quiz** If  $z = e^{-y} \sin(x)$ , which of the following is  $z_{xx} + z_{yy}$ ?

- (a)  $e^{-y} \sin(x)$ ,      (b) 0,      (c)  $-e^{-y} \sin(x)$ ,      (d)  $-e^{-y} \cos(x)$ .

## 4. Quiz on Partial Derivatives

Choose the correct option for each of the following.

### Begin Quiz

1. If  $z = x^2 + 3xy + y^3$  then  $\frac{\partial z}{\partial x}$  is

(a)  $2x + 3y + 3y^2$ ,

(b)  $2x + 3x + 3y^2$ ,

(c)  $2x + 3x$ ,

(d)  $2x + 3y$ .

2. If  $w = 1/r$ , where  $r^2 = x^2 + y^2 + z^2$ , then  $xw_x + yw_y + zw_z$  is

(a)  $-1/r$ ,

(b)  $1/r$ ,

(c)  $-1/r^2$ ,

(d)  $1/r^2$ .

3. If  $u = \sqrt{\frac{x}{y}}$  then  $u_{xx}$  is

(a)  $-\frac{1}{4\sqrt{y^3x^3}}$ ,

(b)  $-\frac{1}{4\sqrt{yx^3}}$ ,

(c)  $-\frac{1}{8\sqrt{y^3x^3}}$ ,

(d)  $-\frac{1}{8\sqrt{yx^3}}$ .

### End Quiz

## Solutions to Exercises

**Exercise 1(a)** To calculate the partial derivative  $\frac{\partial z}{\partial x}$  of the function  $z = x^2y^4$ , the factor  $y^4$  is treated as a constant:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} (x^2y^4) = \frac{\partial}{\partial x} (x^2) \times y^4 = 2x^{(2-1)} \times y^4 = 2xy^4.$$

Similarly, to find the partial derivative  $\frac{\partial z}{\partial y}$ , the factor  $x^2$  is treated as a constant:

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} (x^2y^4) = x^2 \times \frac{\partial}{\partial y} (y^4) = x^2 \times 4y^{(4-1)} = 4x^2y^3.$$

Click on the green square to return



**Exercise 1(b)** To calculate  $\frac{\partial z}{\partial x}$  for the function  $z = (x^4 + x^2)y^3$ , the factor  $y^3$  is treated as a constant:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} ((x^4 + x^2)y^3) = \frac{\partial}{\partial x} (x^4 + x^2) \times y^3 = (4x^3 + 2x)y^3.$$

To find the partial derivative  $\frac{\partial z}{\partial y}$  the factor  $(x^4 + x^2)$  is treated as a constant:

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} ((x^4 + x^2)y^3) = (x^4 + x^2) \times \frac{\partial}{\partial y} y^3 = 3(x^4 + x^2)y^2.$$

Click on the green square to return



**Exercise 1(c)** If  $z = y^{\frac{1}{2}} \sin(x)$  then to calculate  $\frac{\partial z}{\partial x}$  the  $y^{\frac{1}{2}}$  factor is kept constant:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} \left( y^{\frac{1}{2}} \sin(x) \right) = y^{\frac{1}{2}} \times \frac{\partial}{\partial x} (\sin(x)) = y^{\frac{1}{2}} \cos(x).$$

Similarly, to evaluate the partial derivative  $\frac{\partial z}{\partial y}$  the factor  $\sin(x)$  is treated as a constant:

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} \left( y^{\frac{1}{2}} \sin(x) \right) = \frac{\partial}{\partial y} y^{\frac{1}{2}} \times \sin(x) = \frac{1}{2} y^{-\frac{1}{2}} \sin(x).$$

Click on the green square to return





**Exercise 2(a)** The function  $z = (x^2 + 3x) \sin(y)$  can be written as  $z = uv$ , where  $u = (x^2 + 3x)$  and  $v = \sin(y)$ . The partial derivatives of  $u$  and  $v$  with respect to the variable  $x$  are

$$\frac{\partial u}{\partial x} = 2x + 3, \quad \frac{\partial v}{\partial x} = 0,$$

while the partial derivatives with respect to  $y$  are

$$\frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial y} = \cos(y).$$

Applying the *product rule*

$$\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x} v + u \frac{\partial v}{\partial x} = (2x + 3) \sin(y).$$

$$\frac{\partial z}{\partial y} = \frac{\partial u}{\partial y} v + u \frac{\partial v}{\partial y} = (x^2 + 3x) \cos(y).$$

Click on the green square to return



**Exercise 2(b)**

The function  $z = \frac{\cos(x)}{y^5}$  can be written as  $z = \cos(x)y^{-5}$ .

Treating the factor  $y^{-5}$  as a constant and differentiating with respect to  $x$ :

$$\frac{\partial z}{\partial x} = -\sin(x)y^{-5} = -\frac{\sin(x)}{y^5}.$$

Treating  $\cos(x)$  as a constant and differentiating with respect to  $y$ :

$$\frac{\partial v}{\partial y} = \cos(x)(-5y^{-6}) = -5\frac{\cos(x)}{y^6}.$$

Click on the green square to return



**Exercise 2(c)** The function  $z = \ln(xy)$  can be rewritten as (see the package on [logarithms](#))

$$z = \ln(xy) = \ln(x) + \ln(y).$$

Thus the partial derivative of  $z$  with respect to  $x$  is

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(\ln(x) + \ln(y)) = \frac{\partial}{\partial x} \ln(x) = \frac{1}{x}.$$

Similarly the partial derivative of  $z$  with respect to  $y$  is

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(\ln(x) + \ln(y)) = \frac{\partial}{\partial y} \ln(y) = \frac{1}{y}.$$

[Click on the green square to return](#)



**Exercise 2(d)** To calculate the partial derivatives of the function  $z = \sin(x) \cos(xy)$  the *product rule* has to be applied

$$\begin{aligned}\frac{\partial z}{\partial x} &= \cos(xy) \frac{\partial}{\partial x} \sin(x) + \sin(x) \frac{\partial}{\partial x} \cos(xy), \\ \frac{\partial z}{\partial y} &= \cos(xy) \frac{\partial}{\partial y} \sin(x) + \sin(x) \frac{\partial}{\partial y} \cos(xy).\end{aligned}$$

Using the *chain rule* with  $u = xy$  for the partial derivatives of  $\cos(xy)$

$$\begin{aligned}\frac{\partial}{\partial x} \cos(xy) &= \frac{\partial \cos(u)}{\partial u} \frac{\partial u}{\partial x} = -\sin(u)y = -y \sin(xy), \\ \frac{\partial}{\partial y} \cos(xy) &= \frac{\partial \cos(u)}{\partial u} \frac{\partial u}{\partial y} = -\sin(u)x = -x \sin(xy).\end{aligned}$$

Thus the partial derivatives of  $z = \sin(x) \cos(xy)$  are

$$\frac{\partial z}{\partial x} = \cos(xy) \cos(x) - y \sin(x) \sin(xy), \quad \frac{\partial z}{\partial y} = -x \sin(x) \sin(xy).$$

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**Exercise 2(e)** To calculate the partial derivatives of  $z = e^{(x^2+y^2)}$  the *chain rule* has to be applied with  $u = (x^2 + y^2)$ :

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial}{\partial u} (e^u) \frac{\partial u}{\partial x} = e^u \frac{\partial u}{\partial x}, \\ \frac{\partial z}{\partial y} &= \frac{\partial}{\partial u} (e^u) \frac{\partial u}{\partial y} = e^u \frac{\partial u}{\partial y}.\end{aligned}$$

The partial derivatives of  $u = (x^2 + y^2)$  are

$$\frac{\partial u}{\partial x} = \frac{\partial(x^2)}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = \frac{\partial(y^2)}{\partial y} = 2y.$$

Therefore the partial derivatives of the function  $z = e^{(x^2+y^2)}$  are

$$\begin{aligned}\frac{\partial z}{\partial x} &= e^u \frac{\partial u}{\partial x} = 2x e^{(x^2+y^2)}, \\ \frac{\partial z}{\partial y} &= e^u \frac{\partial u}{\partial y} = 2y e^{(x^2+y^2)}.\end{aligned}$$

Click on the green square to return



**Exercise 2(f)** Applying the *chain rule* with  $u = x^2 + y$  the partial derivatives of the function  $z = \sin(x^2 + y)$  can be written as

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial}{\partial u} (\sin(u)) \frac{\partial u}{\partial x} = \cos(u) \frac{\partial u}{\partial x}, \\ \frac{\partial z}{\partial y} &= \frac{\partial}{\partial u} (\sin(u)) \frac{\partial u}{\partial y} = \cos(u) \frac{\partial u}{\partial y}.\end{aligned}$$

The partial derivatives of  $u = x^2 + y$  are

$$\frac{\partial u}{\partial x} = \frac{\partial x^2}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = \frac{\partial y}{\partial y} = 1.$$

Thus the partial derivatives of the function  $z = \sin(x^2 + y)$  are

$$\begin{aligned}\frac{\partial z}{\partial x} &= \cos(u) \frac{\partial u}{\partial x} = 2x \cos(x^2 + y), \\ \frac{\partial z}{\partial y} &= \cos(u) \frac{\partial u}{\partial y} = \cos(x^2 + y).\end{aligned}$$

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**Exercise 3(a)**

From **exercise 2(a)**, the first order partial derivatives of  $z = (x^2 + 3x) \sin(y)$  are

$$\frac{\partial z}{\partial x} = (2x + 3) \sin(y), \quad \frac{\partial z}{\partial y} = (x^2 + 3x) \cos(y).$$

The *mixed* second order derivatives are

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} ((x^2 + 3x) \cos(y)) = (2x + 3) \cos(y),$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial y} ((2x + 3) \sin(y)) = (2x + 3) \cos(y).$$

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**Exercise 3(b)**

From **exercise 2(b)**, the first order partial derivatives of  $z = \frac{\cos(x)}{y^5}$  are

$$\frac{\partial z}{\partial x} = -\frac{\sin(x)}{y^5}, \quad \frac{\partial z}{\partial y} = -5\frac{\cos(x)}{y^6},$$

so the *mixed* second order derivatives are

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} \left( -5\frac{\cos(x)}{y^6} \right) = 5\frac{\sin(x)}{y^6},$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial y} \left( -\frac{\sin(x)}{y^5} \right) = 5\frac{\sin(x)}{y^6}.$$

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**Exercise 3(c)**

From **exercise 2(c)**, the first order partial derivatives of  $z = \ln(xy)$  are

$$\frac{\partial z}{\partial x} = \frac{1}{x}, \quad \frac{\partial z}{\partial y} = \frac{1}{y}.$$

The *mixed* second order derivatives are

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} \left( \frac{1}{y} \right) = 0,$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial y} \left( \frac{1}{x} \right) = 0.$$

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**Exercise 3(d)** From **exercise 2(d)**, the first order partial derivatives of  $z = \sin(x) \cos(xy)$  are

$$\frac{\partial z}{\partial x} = \cos(x) \cos(xy) - y \sin(x) \sin(xy), \quad \frac{\partial z}{\partial y} = -x \sin(x) \sin(xy).$$

The *mixed* second order derivatives are

$$\begin{aligned} \frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} (-x \sin(x) \sin(xy)) \\ &= -\sin(x) \sin(xy) - x \cos(x) \sin(xy) - xy \sin(x) \cos(xy), \\ \frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial y} (\cos(x) \cos(xy) - y \sin(x) \sin(xy)) \\ &= -x \cos(x) \sin(xy) - \sin(x) \sin(xy) - xy \sin(x) \cos(xy). \end{aligned}$$

**N.B.** In the solution above a *product of three functions* has been differentiated. This can be done by using two applications of the *product rule*.

Click on the green square to return



**Exercise 3(e)** From **exercise 2(e)**, the first order partial derivatives of  $z = e^{(x^2+y^2)}$  are

$$\frac{\partial z}{\partial x} = 2xe^{(x^2+y^2)}, \quad \frac{\partial z}{\partial y} = 2ye^{(x^2+y^2)}.$$

The *mixed* second order derivatives are thus

$$\begin{aligned} \frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} \left( 2ye^{(x^2+y^2)} \right) = 4xye^{(x^2+y^2)}, \\ \frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial y} \left( 2xe^{(x^2+y^2)} \right) = 4yxe^{(x^2+y^2)}. \end{aligned}$$

Click on the green square to return



**Exercise 3(f)** From **exercise 2(f)**, the first order partial derivatives of  $z = \sin(x^2 + y)$  are

$$\frac{\partial z}{\partial x} = 2x \cos(x^2 + y), \quad \frac{\partial z}{\partial y} = \cos(x^2 + y).$$

The *mixed* second order derivatives are thus

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} (\cos(x^2 + y)) = -2x \sin(x^2 + y),$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial y} (2x \cos(x^2 + y)) = -2x \sin(x^2 + y).$$

Click on the green square to return



## Solutions to Quizzes

### Solution to Quiz:

To determine which of the options is correct, the partial derivatives of  $z = \cos(xy)$  must be calculated. From the calculations of **exercise 2(d)** the partial derivatives of  $z = \cos(xy)$  are

$$\begin{aligned}\frac{\partial}{\partial x} \cos(xy) &= \frac{\partial \cos(u)}{\partial u} \frac{\partial u}{\partial x} = -\sin(u)y = -y \sin(xy), \\ \frac{\partial}{\partial y} \cos(xy) &= \frac{\partial \cos(u)}{\partial u} \frac{\partial u}{\partial y} = -\sin(u)x = -x \sin(xy).\end{aligned}$$

Therefore

$$\frac{1}{y} \frac{\partial}{\partial x} \cos(xy) = -\sin(xy) = \frac{1}{x} \frac{\partial}{\partial y} \cos(xy).$$

The other choices, if checked, will be found to be false.

**End Quiz**

**Solution to Quiz:**

The first order derivatives of  $z = e^{-y} \sin(x)$  are

$$z_x = e^{-y} \cos(x), \quad z_y = -e^{-y} \sin(x),$$

where  $e^{-y}$  is kept constant for the first differentiation and  $\sin(x)$  for the second. Continuing in this way, the second order derivatives  $z_{xx}$  and  $z_{yy}$  are given by the expressions

$$\begin{aligned} z_{xx} &= \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} (e^{-y} \cos(x)) = -e^{-y} \sin(x), \\ z_{yy} &= \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial y} (-e^{-y} \sin(x)) = e^{-y} \sin(x). \end{aligned}$$

Adding these two equations together gives

$$z_{xx} + z_{yy} = 0.$$

End Quiz