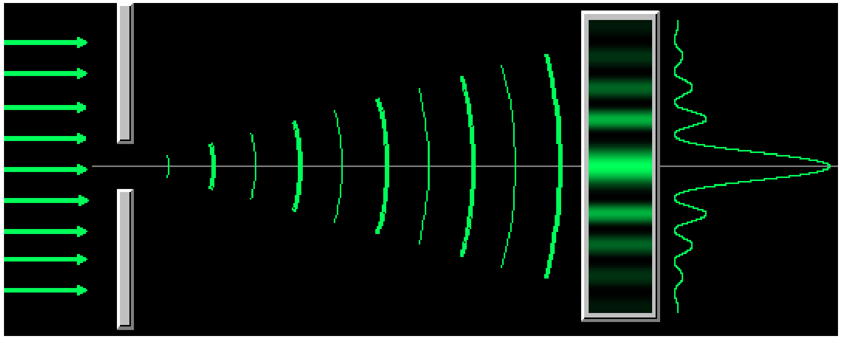
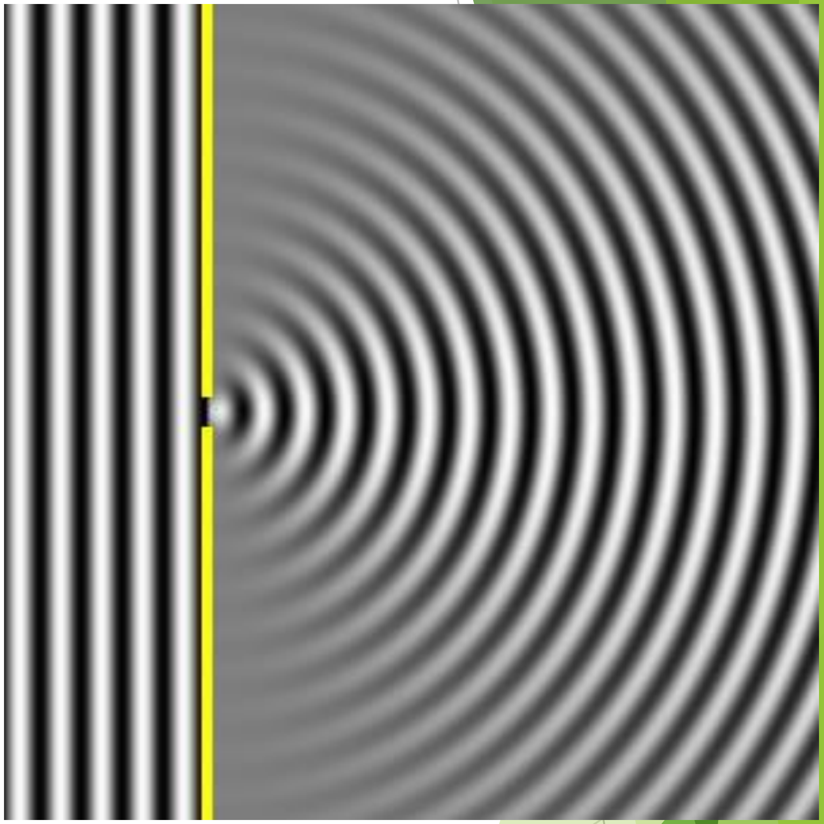
Light passage through a single slit





Lookangmany thanks to Fu-Kwun Hwang and author of Easy Java Simulation = Francisco Esquembre / CC BY-SA

Huygens' principle – every point on a wave front is considered to be a secondary source of spherical wavelets.

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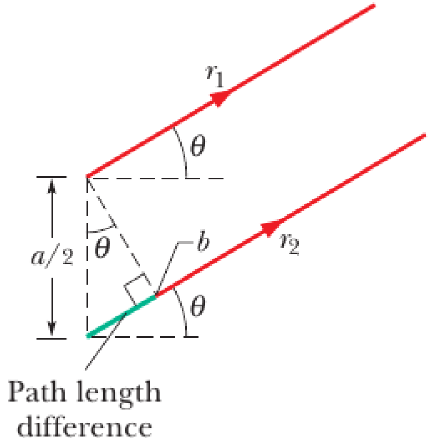
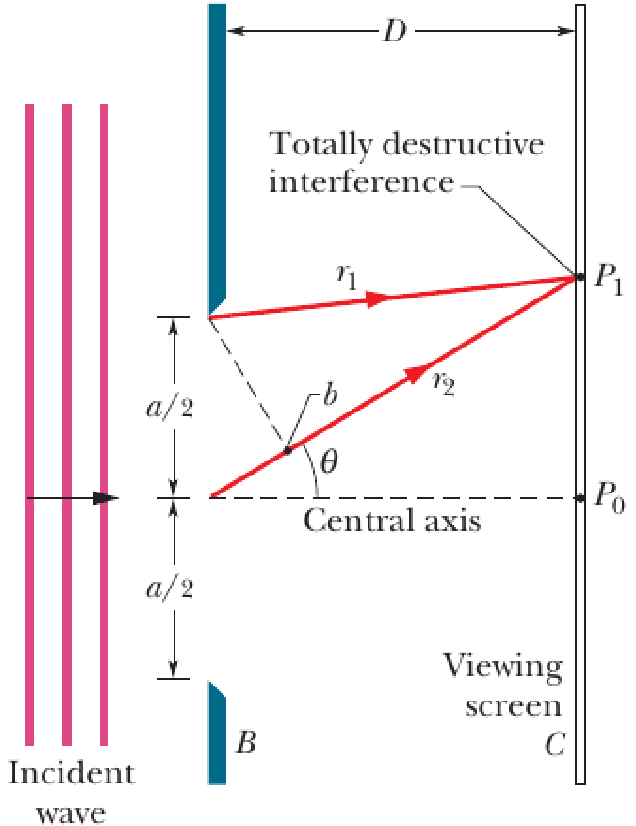


This path length difference shifts one wave from the other, which determines the interference.

This pair of rays cancel each other at P_1. So do all such pairings.



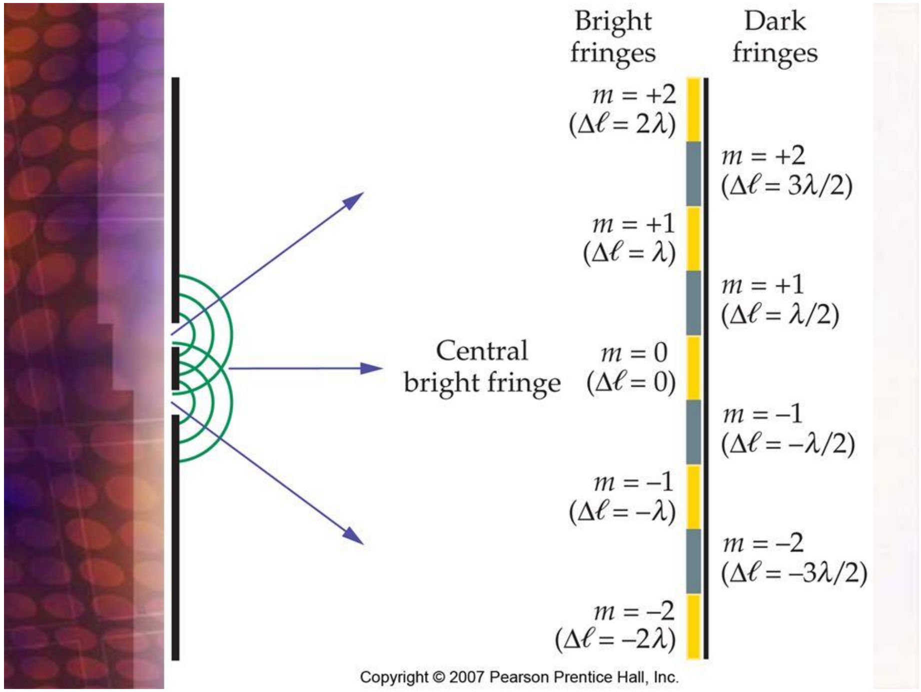
For $D >> a$, we can approximate rays r_1 and r_2 as being parallel, at angle θ to the central axis. The path length difference is then $\frac{a}{2}\sin(\theta)$. The condition for destructive interference is

$\frac{a}{2}\sin(\theta) = \frac{\lambda}{2}$. By repeating a similar analysis we find that the minima are found by

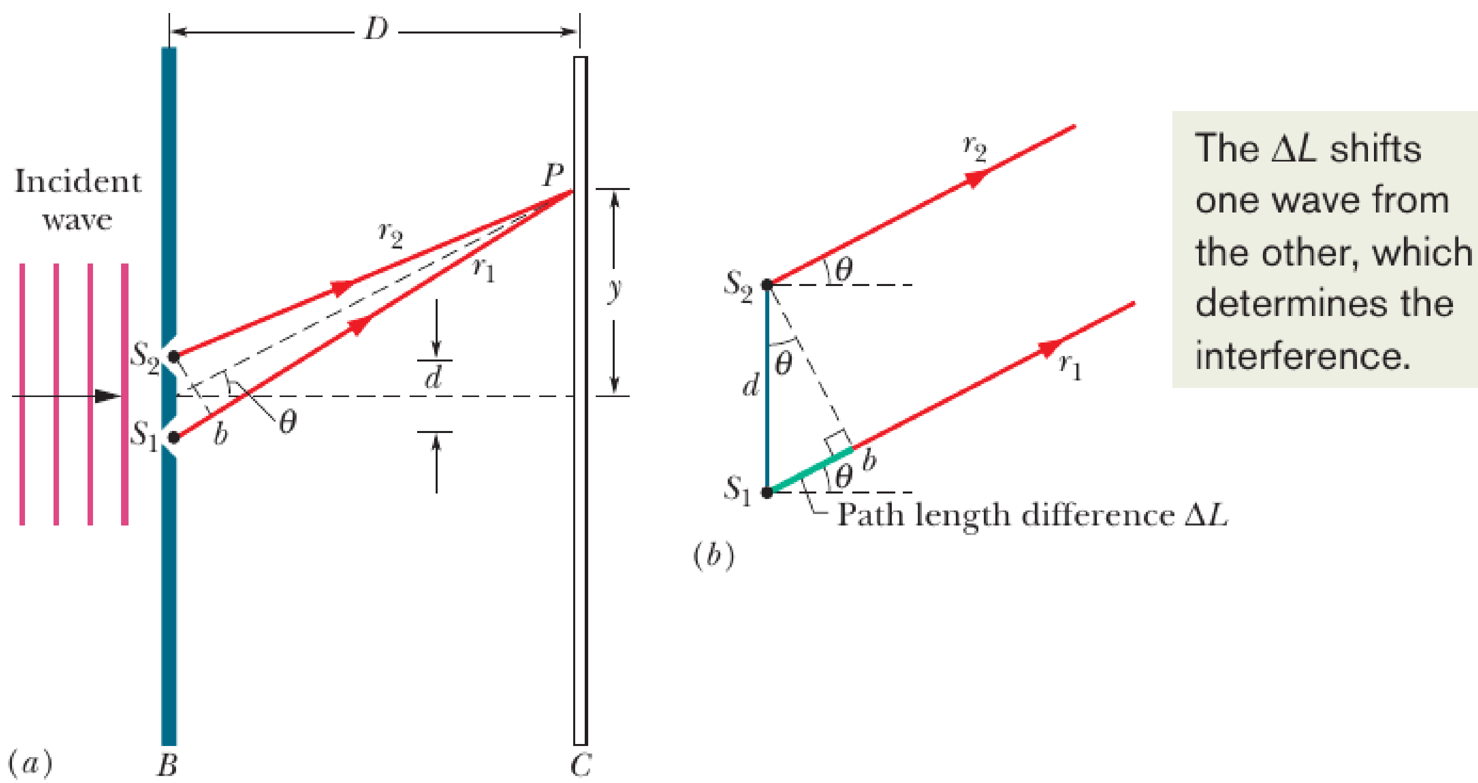
$$\sin(\theta) = \frac{m\lambda}{a}$$

Double slit diffraction

Huygens' principle – every point on a wave front is considered to be a secondary source of spherical wavelets.



Double slit diffraction



Qualitative explanation

- Let's assume that the electric field produced by waves originating at different points is in the same direction.
- $\vec{E} = E\,\vec{e} = \vec{E}_1 + \vec{E}_2 = E_1\,\vec{e}_1 + E_2\vec{e}_2$
- Assuming that both fields are in the same direction the intensity is proportional to
- $I \propto |\mathrm{E}|^2 = (E_1 + E_2)(E_1^* + E_2^*) = |E_1|^2 + |E_2|^2 + E_1^* E_2 + E_1 E_2^*$
- Far from the slit, each wave may be considered as a plane wave $E_1 = A_1 e^{i\phi_1(x)}; E_2 = A_2 e^{i\phi_2(x)}$ and for equal amplitudes

$$I \propto 2A_1^2\left(1 + \cos\left(\phi_1(x) - \phi_2(x)\right)\right) = 2A_1^2\left(1 + \cos\left(\Delta\phi(x)\right)\right)$$

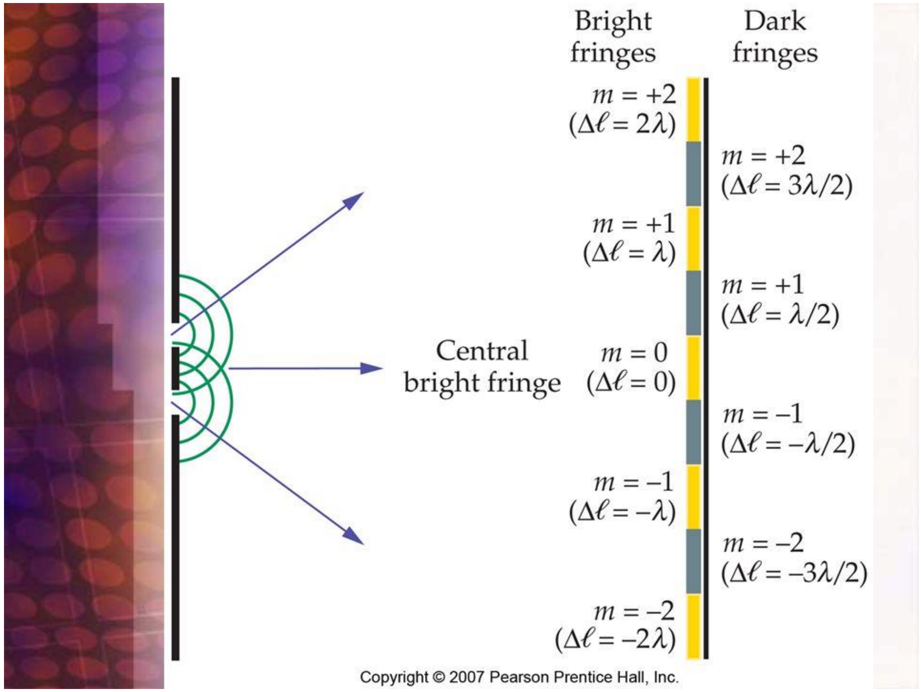
$$= 4A_1^2 \cos^2\left(\frac{\Delta\phi(x)}{2}\right)$$

Qualitative explanation

$$I \propto 4A_1^2 \cos^2\left(\frac{\Delta\phi(x)}{2}\right)$$

The intensity is maximal when the phase difference is equal to $2\pi n$

(for any integer n)



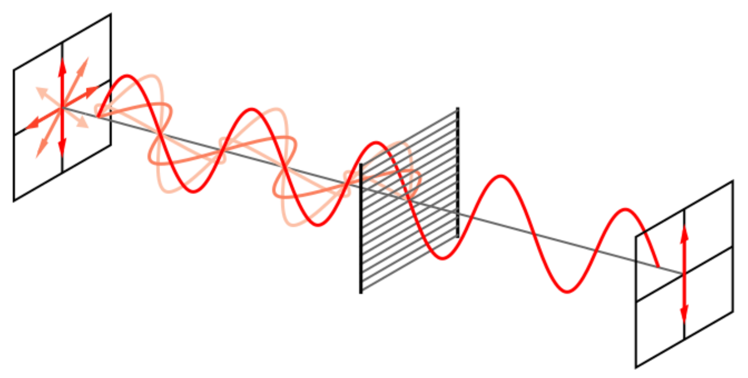
What happens when photons are passing one at a time?

- A particle is expected to go through a specific point in the slit. How can it interfere with itself?
- Under these conditions, each photon hits another location and the average intensity (of many single photons) results in the same diffraction image as many simultaneous photons.
- The same behavior was observed for electrons and is not limited to photons.

Polarization

- The electric field due to a linearly polarized light (in the x-y plane) may be written as

$$\vec{E}(z,t) = E_0\vec{e}_p\cos(kz-\omega t)$$



- The unit vector of the polarization direction may be written as

$$\vec{e}_p = \cos(\phi)\ \hat{x} + \sin(\phi)\ \hat{y} = \cos(\phi)|x\rangle + \sin(\phi)\,|y\rangle$$

Polarization

- When the light is filtered by a polarizer, which only allows the field in the *x* direction to pass through, the intensity becomes $I \propto |E|^2 \propto E_0^2 \cos^2(\phi)$
- When photons hit the polarizer, one at a time, some photons may pass and others won't and after a large number of photons, *N*, hit the polarizer the average number of passing photons is proportional to $Ncos^2(\phi)$

Group and phase velocities

Consider a wave which is the solution of the classical wave equation:

$$\frac{\partial^2 \phi}{\partial t^2} = v^2 \frac{\partial^2 \phi}{\partial x^2}$$

As we already know the solution is of the form:

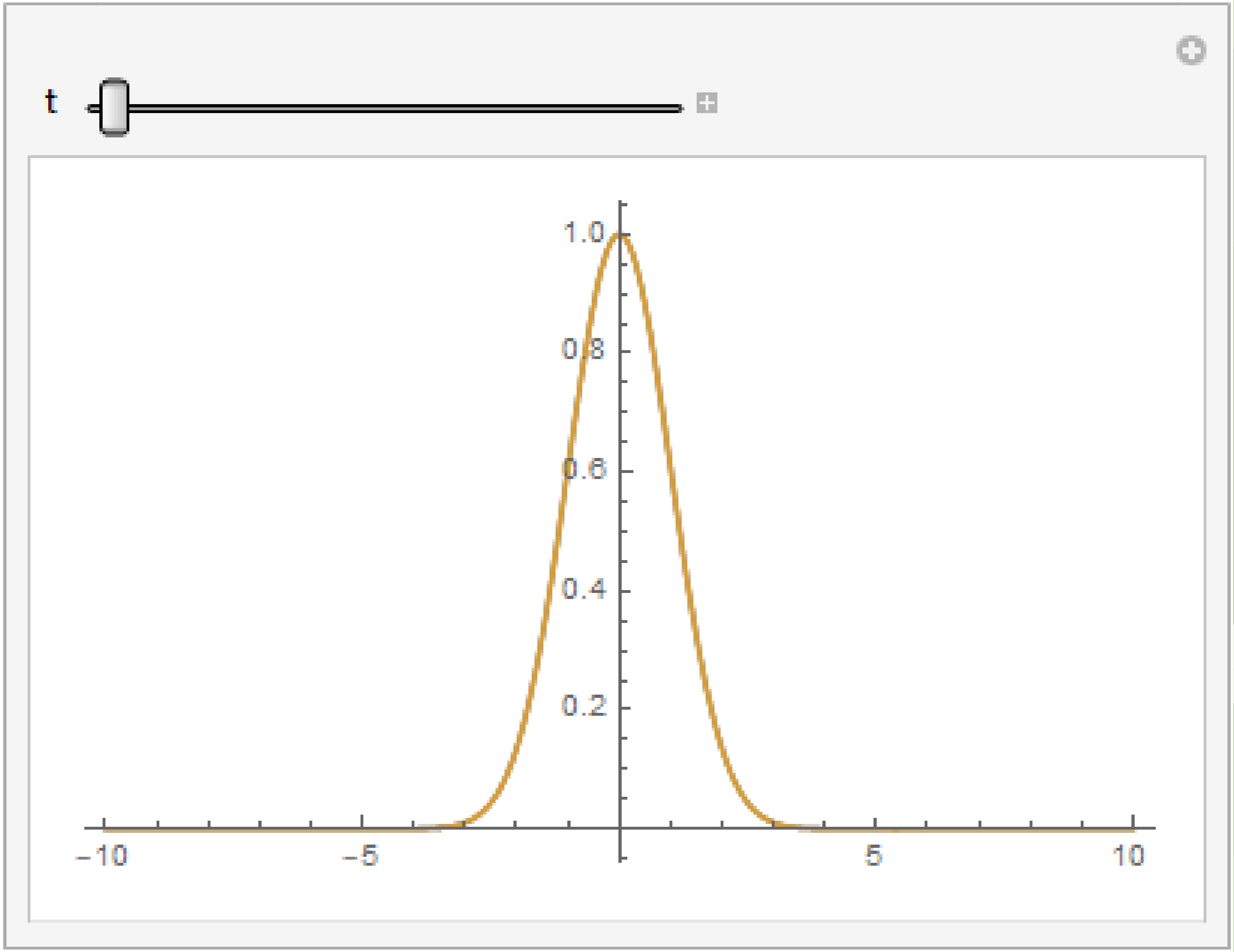
$$\phi = Acos\big(k(x - vt)\big) = Acos(kx - \omega t)$$

Where $\omega = kv$. Note that in this case, $\frac{\omega}{k} = \frac{d\omega}{dk} = v$. So waves of all wavelengths and frequencies travel with the same speed and, therefore, any linear combination of waves travels with the same speed.

For example, if we consider a Gaussian wavepacket (representing a localized pulse) of the form:

$$\phi(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi/a^2}} e^{-\frac{a^2k^2}{2}} e^{ik(x-vt)} dk = e^{-\frac{(x-vt)^2}{2a^2}}$$

Group and phase velocities



The sum of waves with different wavenumbers, frequencies and velocities

Consider the sum of two waves:

$$\phi_1(x,t) = Acos\big(k_1(x - v_1 t)\big) \qquad v_1 = \omega_1/k_1$$

$$\phi_2(x,t) = Acos\big(k_2(x - v_2 t)\big) \qquad v_2 = \omega_2/k_2$$

$$\phi(x,t) = \phi_1(x,t) + \phi_2(x,t) = Acos\big(k_1(x - v_1 t)\big) + Acos\big(k_2(x - v_2 t)\big)$$

$$= 2Acos\left(\frac{1}{2}k_1(x - v_1 t) + \frac{1}{2}k_2(x - v_2 t)\right)\cos\left(\frac{1}{2}k_1(x - v_1 t) - \frac{1}{2}k_2(x - v_2 t)\right)$$

$$= 2Acos\left(\frac{1}{2}(k_1 + k_2)x - \frac{1}{2}(\omega_1 + \omega_2)t\right)cos(\Delta kx - \Delta\omega t)$$

Where

$$\Delta k = \frac{1}{2}(k_1 - k_2) \quad and \quad \Delta\omega = \frac{1}{2}(\omega_1 - \omega_2)$$

The sum of waves with different wavenumbers, frequencies and velocities

Let's define $k_1 = k_0 + \Delta k; k_2 = k_0 - \Delta k; \omega_1 = \omega_0 + \Delta\omega; \omega_2 = \omega_0 - \Delta\omega$

Which allows us to write:

$$\phi(x,t) = 2Acos(k_0 x - \omega_0 t)cos(\Delta kx - \Delta\omega t)$$

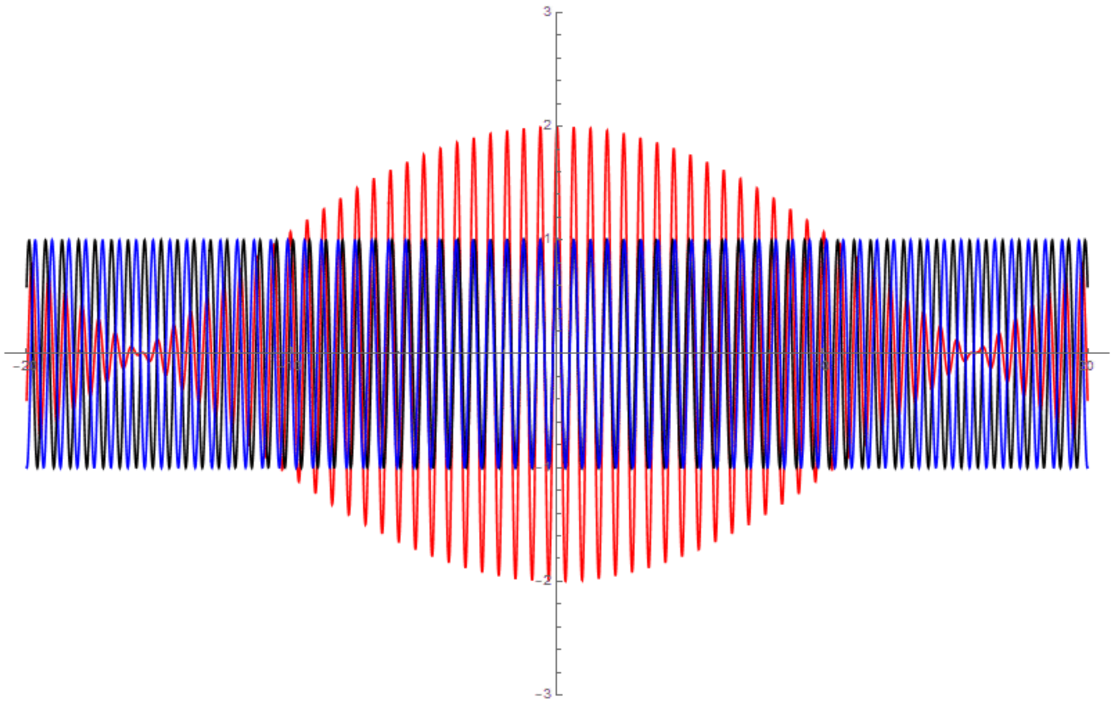
Assuming that $\Delta k \ll k_0$ and $\Delta\omega \ll \omega_0$ we notice that we have a modulated

amplitude $2Acos\left(\Delta k\left(x - \frac{\Delta\omega}{\Delta k}t\right)\right)$ of the wave $cos\left(k_0\left(x - \frac{\omega_0}{k_0}t\right)\right)$.

The second term is simply a wave with almost the same frequency and wavelength of the original waves (and therefore, $v_p = \omega_0/k_0$) and the envelope (the modulated amplitude) is a wave moving with velocity $v_g = \frac{\Delta\omega}{\Delta k}$ with the limit:

$\lim_{\Delta\omega\to 0,\Delta k\to 0} v_g = \frac{\partial\omega}{\partial k}$. The generalization to 3D is simple: $\vec{v}_g = \vec{\nabla}_k \omega(\vec{k})$

The sum of waves with different wavenumbers, frequencies and velocities



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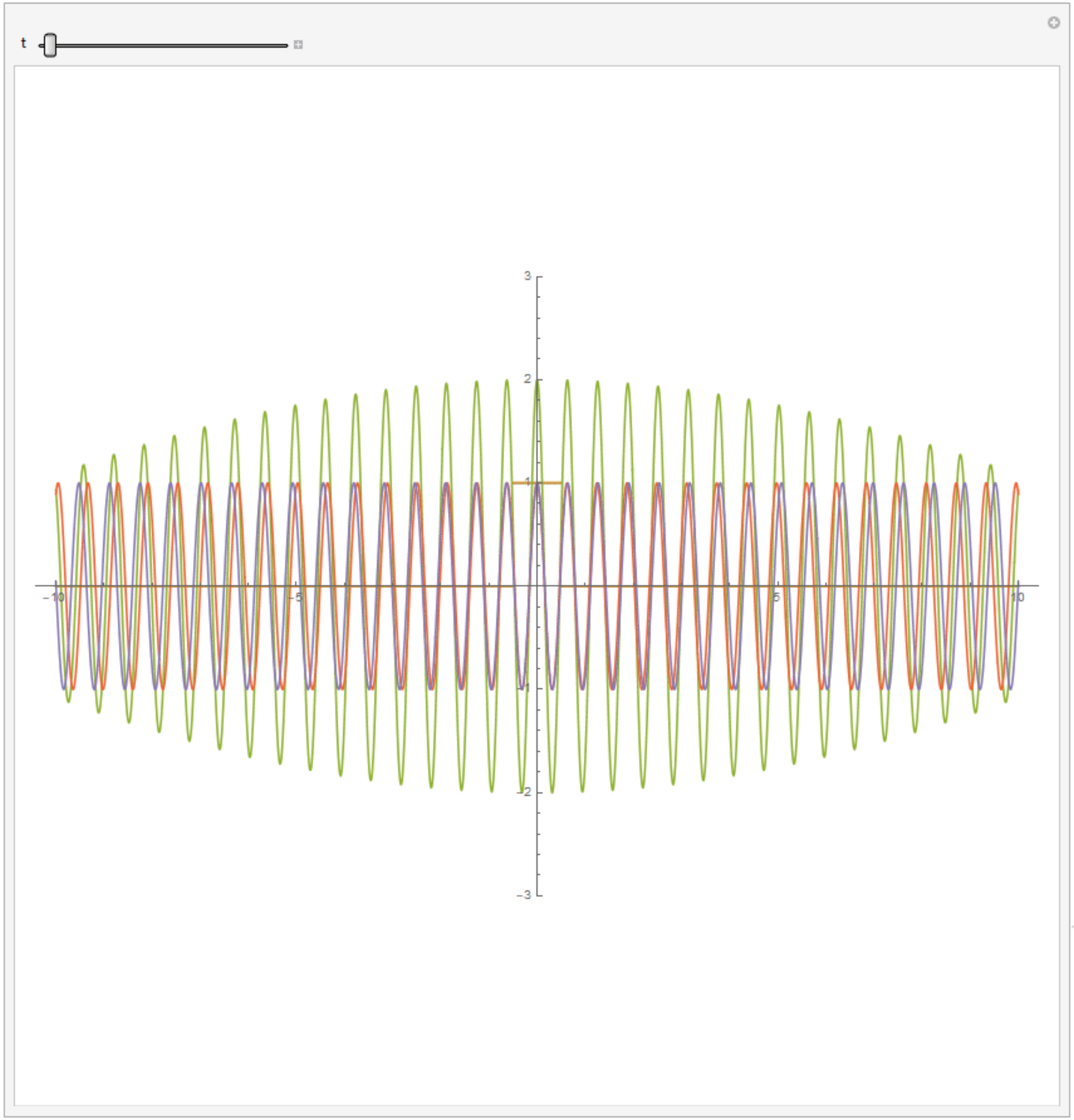
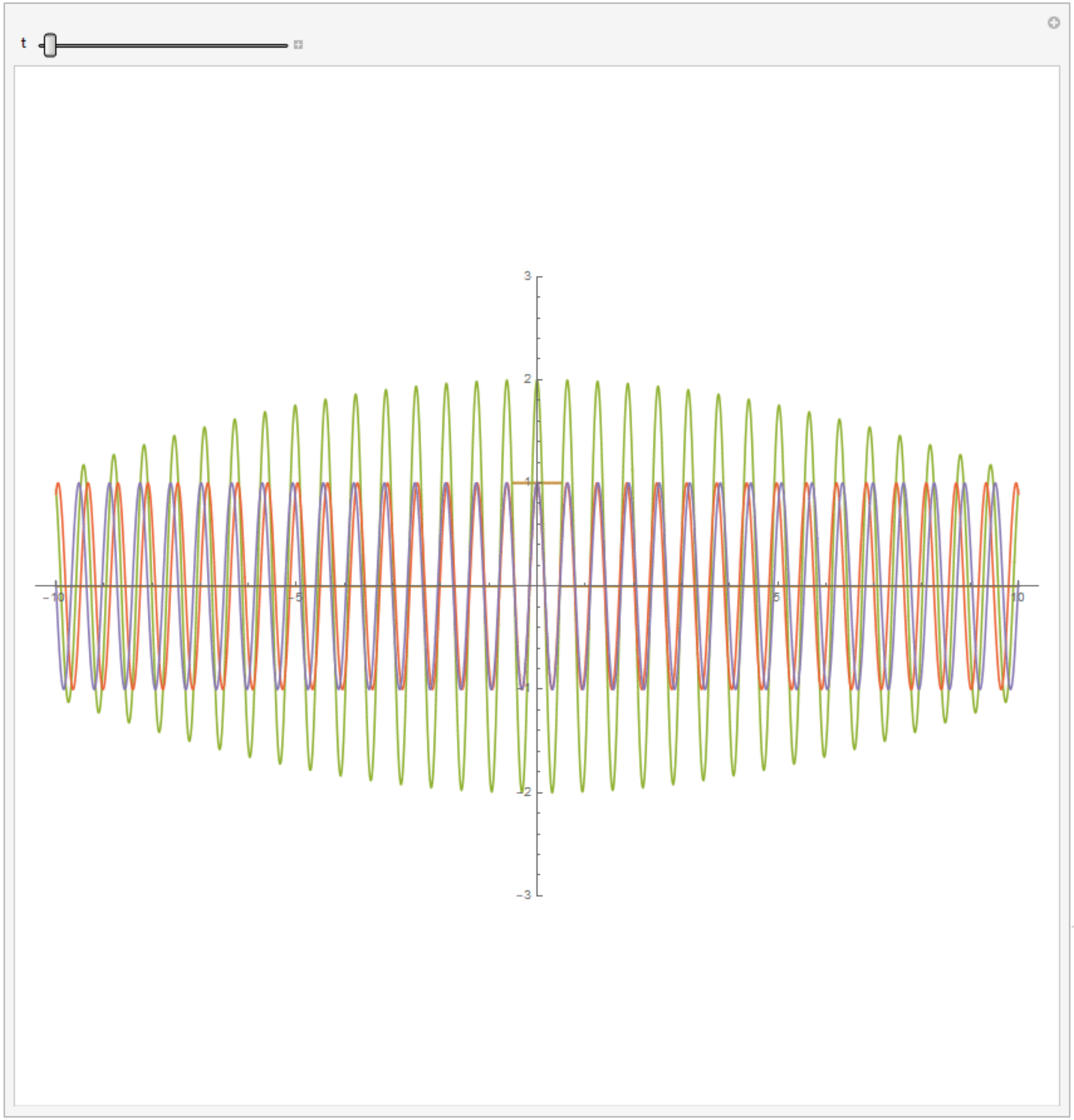


The sum of waves with different wavenumbers, frequencies and velocities



The sum of many waves

For the general case, of a wavepacket

$$\phi(\vec{r},t) = \int_{-\infty}^{\infty} \tilde{\phi}(\vec{k}) e^{i(\vec{k}\cdot\vec{r}-\omega t)} d\vec{k}$$

If we consider $\tilde{\phi}(\vec{k})$ which is nearly monochromatic, i.e., sharply centered around $\vec{k}_0$, we may expand $\vec{k}$ and $\omega(\vec{k})$ as:

$$\vec{k} = \vec{k}_0 + (\vec{k} - \vec{k}_0)$$

$$\omega = \omega(\vec{k}_0) + \vec{\nabla}_k \omega(\vec{k}) \Big|_{\vec{k}=\vec{k}_0} (\vec{k} - \vec{k}_0) + O\left((\vec{k} - \vec{k}_0)^2\right)$$

The sum of many waves

The wavepacket takes the form of a wave modulated by an envelope with velocity

$$\vec{v}_g = \vec{\nabla}_k \omega(\vec{k})|_{\vec{k}=\vec{k}_0}$$

$$\phi(\vec{r},t) = e^{i(\vec{k}_0\cdot\vec{r}-\omega(\vec{k}_0)t)} \int_{-\infty}^{\infty} \tilde{\phi}(\vec{k}) e^{i\left(\left((\vec{k}-\vec{k}_0)\right)\cdot\vec{r}-\nabla_k\omega(\vec{k})|_{\vec{k}=\vec{k}_0}(\vec{k}-\vec{k}_0)t\right)} d\vec{k}$$

Wave with the central wavevector and the corresponding frequency.

$$\vec{v}_p = \hat{k}_0 \frac{\omega(\vec{k}_0)}{|\vec{k}_0|}$$

Envelope traveling with the group velocity

$$\vec{v}_g = \vec{\nabla}_k \omega(\vec{k})|_{\vec{k}=\vec{k}_0}$$

Fourier transform

The Fourier transform is defined as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} dk$$

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

The definitions above ensure that applying the transform and its inverse results in the identity operation.

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x') e^{-ikx'} dx' e^{ikx} dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x') e^{-ik(x'-x)} dx' dk = \int_{-\infty}^{\infty} f(x')\delta(x-x')\, dx' = f(x)$$

Fourier transform

We used the definition of a delta function

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx} dk$$

The delta function is zero if its argument is not zero and diverges for a zero argument such that

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$$

Examples Fourier Transform

- Find the FT of the following function:

$$s(t) = e^{-u|t|} \qquad u > 0$$

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$$F\{s(t)\} = \tilde{s}(\omega) = \int_{-\infty}^{\infty} s(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} e^{-(i\omega t + u|t|)}dt$$

$$= 2Re\left[\int_{0}^{\infty} e^{-(i\omega + u)t}dt\right] = 2Re\left[-\frac{e^{-(i\omega+u)t}}{i\omega + u}\Big|_{0}^{\infty}\right]$$

$$= 2Re\left[\frac{1}{i\omega + u}\right] = 2Re\left[\frac{-i\omega + u}{\omega^2 + u^2}\right] = 2\frac{u}{\omega^2 + u^2}$$

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$$= \int_{-\infty}^{\infty} e^{-(i\omega+ia)t}dt = 2\pi\delta(\omega + a\ )$$