Quantum Mechanics 3 - Class Exercise 6

14.12.2022

Question 1

Using the propagator of harmonic oscillator, derive the first order correction to the ground state of the anharmonic oscillator with $V(x) = \frac{1}{2}m\omega^2 x^2 + bx^4$, where $b \ll m^2\omega^3/\hbar$.

Solution

Before we dive into the complex algebra, let us try to estimate the answer a priori from dimensional considerations. Since we are looking for first order approximation we know that the answer has to be proportional to b (rather than b^2, b^3, etc). Therefore the answer has to be of the form $\Delta E \propto E_0 b$. However, b is a dimensionful quantity so we need to compare it with something that has the same dimensions in order to obtain a dimensionless quantity. You should convince yourself that $\hbar b/m^2\omega^3$ is dimensionless. Thus, we guess

$$\Delta E \propto E_0 \frac{\hbar b}{m^2\omega^3} \propto \hbar\omega \frac{\hbar b}{m^2\omega^3} = \frac{\hbar^2 b}{m^2\omega^2}. \tag{1}$$

All that has left is to find the dimensionless constant of proportionality. This will require us to do "a little" algebra so we divide the solution into 6 steps.

Step 1: find the relation between E_0 and K

We start with the formula derived in class

$$K(x_f, t_f; x_i, t_i) = \sum_n \psi_n(x_f)\,\psi_n^*(x_i)\,\mathrm{e}^{-iE_n(t_f - t_i)/\hbar}. \tag{2}$$

For convenience, we will set $x_i = x_f = 0$. We will also extend to "imaginary time" and set $t_i = i\beta\hbar/2$, $t_f = -i\beta\hbar/2$. Thus

$$K(0, -i\beta\hbar/2; 0, i\beta\hbar/2) = \sum_n |\psi_n(0)|^2\,\mathrm{e}^{-\beta E_n} = \mathrm{e}^{-\beta E_0}\sum_n |\psi_n(0)|^2\,\mathrm{e}^{-\beta(E_n - E_0)}. \tag{3}$$

Next, in order to extract E_0 we would like to apply logarithm on both sides of Eq. (3). However, ln can only be applied on dimensionless quantities but the propagator has dimensions of inverse length in 1D (explain this!). Therefore, before applying ln we need to multiply both sides of the equation with some quantity with units of length. Such quantity can be for example $\sqrt{\hbar/m\omega}$. Thus

$$E_0 = -\frac{1}{\beta}\ln\left(\sqrt{\frac{\hbar}{m\omega}}K(0, -i\beta\hbar/2; 0, i\beta\hbar/2)\right) + \frac{1}{\beta}\ln\left(\sum_n \sqrt{\frac{\hbar}{m\omega}}\,|\psi_n(0)|^2\,\mathrm{e}^{-\beta(E_n - E_0)}\right). \tag{4}$$

By taking the limit $\beta \to \infty$ it is easy to see that the second term on the RHS of Eq. (4) vanishes. Therefore we only end up with

$$\boxed{E_0 = -\lim_{\beta\to\infty} \frac{1}{\beta} \ln\left(\sqrt{\frac{\hbar}{m\omega}} K\left(0, -i\beta\hbar/2; 0, i\beta\hbar/2\right)\right)}. \tag{5}$$

Thus, the problem of finding E_0 is equivalent to calculating the propagator at imaginary times and taking β to infinity.

Step 2: express K as functional derivatives of K_J

The propagator is calculated via the path integral,

$$K\left(0, -i\beta\hbar/2; 0, i\beta\hbar/2\right) = \int_{x(t_i=i\beta\hbar/2)=0}^{x\left(t_f=-i\beta\hbar/2\right)=0} \mathcal{D}x\left(t\right) \exp\left[\frac{i}{\hbar}\int_{i\beta\hbar/2}^{-i\beta\hbar/2}\left(\frac{1}{2}m\dot{x}^2 - V\left(x\right)\right)dt\right]. \tag{6}$$

Since the end points are expressed with imaginary times $\pm i\beta\hbar/2$, let us change variables accordingly and define $\tau \equiv it$. Note that the kinetic term switches sign as $\dot{x}^2 = \left(dx/dt\right)^2 = -\left(dx/d\tau\right)^2$. The propagator can then be written as

$$\begin{aligned} K\left(0, -i\beta\hbar/2; 0, i\beta\hbar/2\right) &= \int_{x(\tau_i=-\beta\hbar/2)=0}^{x\left(\tau_f=\beta\hbar/2\right)=0} \mathcal{D}x\left(\tau\right) \exp\left[-\frac{1}{\hbar}\int_{-\beta\hbar/2}^{\beta\hbar/2}\left(\frac{1}{2}m\left(\frac{dx}{d\tau}\right)^2 + V\left(x\right)\right)d\tau\right] \\ &= \int_{x(\tau_i=-\beta\hbar/2)=0}^{x\left(\tau_f=\beta\hbar/2\right)=0} \mathcal{D}x\left(\tau\right) \exp\left[-\frac{1}{\hbar}\int_{-\beta\hbar/2}^{\beta\hbar/2} H\left[x\left(\tau\right)\right]d\tau\right], \end{aligned} \tag{7}$$

where $H\left[x\left(\tau\right)\right]$ is the Hamiltonian expressed with τ as the time parameter:

$$H = \frac{1}{2}m\left(\dot{x}^2 + \omega^2 x^2\right) + bx^4. \tag{8}$$

Note that for brevity I now use $\dot{x} = dx/d\tau$.

We can now separate the b term from the rest of the terms at the exponent, and use the approximation $\mathrm{e}^x \approx 1 + x$ for that term as b is small.

$$K\left(0, -i\beta\hbar/2; 0, i\beta\hbar/2\right) = \int_{x(\tau_i=-\beta\hbar/2)=0}^{x\left(\tau_f=\beta\hbar/2\right)=0} \mathcal{D}x\left(\tau\right) \mathrm{e}^{-\frac{1}{\hbar}\int_{-\beta\hbar/2}^{\beta\hbar/2}\frac{1}{2}m\left(\dot{x}^2+\omega^2x^2\right)d\tau}\left[1 - \frac{b}{\hbar}\int_{-\beta\hbar/2}^{\beta\hbar/2} x^4\left(\tau\right)d\tau\right] \tag{9}$$

In order to calculate that, consider the addition of a source term $J\left(\tau\right)x\left(\tau\right)$ to the Hamiltonian of the harmonic oscillator,

$$K_J\left(0, -i\beta\hbar/2; 0, i\beta\hbar/2\right) \equiv \int_{x(\tau_i=-\beta\hbar/2)=0}^{x\left(\tau_f=\beta\hbar/2\right)=0} \mathcal{D}x\left(\tau\right) \mathrm{e}^{-\frac{1}{\hbar}\int_{-\beta\hbar/2}^{\beta\hbar/2}\left[\frac{1}{2}m\left(\dot{x}^2+\omega^2x^2\right)+Jx\right]d\tau}. \tag{10}$$

As you saw in class, applying functional derivatives on this equation with respect to $J\left(\tau\right)$ is useful since each derivative results an additional $x\left(\tau\right)$ term in the integral. For example:

$$\frac{\delta}{\delta J\left(\tau_1\right)} K_J\left(0, -i\beta\hbar/2; 0, i\beta\hbar/2\right) = \int_{x(\tau_i=-\beta\hbar/2)=0}^{x\left(\tau_f=\beta\hbar/2\right)=0} \mathcal{D}x\left(\tau\right)\left(-\frac{x\left(\tau_1\right)}{\hbar}\right) \mathrm{e}^{-\frac{1}{\hbar}\int_{-\beta\hbar/2}^{\beta\hbar/2}\left[\frac{1}{2}m\left(\dot{x}^2+\omega^2x^2\right)+Jx\right]d\tau}, \tag{11}$$

and therefore

$$\int_{x(\tau_i=-\beta\hbar/2)=0}^{x\left(\tau_f=\beta\hbar/2\right)=0} \mathcal{D}x\left(\tau\right) x\left(\tau_1\right) \mathrm{e}^{-\frac{1}{\hbar}\int_{-\beta\hbar/2}^{\beta\hbar/2}\frac{1}{2}m\left(\dot{x}^2+\omega^2x^2\right)d\tau} = -\hbar\,\frac{\delta}{\delta J\left(\tau_1\right)} K_J\left(0, -i\beta\hbar/2; 0, i\beta\hbar/2\right)\bigg|_{J=0}, \tag{12}$$

Thus, we can write Eq. (9) as

$$\boxed{K\left(0,-i\beta\hbar/2;0,i\beta\hbar/2\right)=\left[1-\hbar^{3}b\int_{-\beta\hbar/2}^{\beta\hbar/2}d\tau_{1}\frac{\delta^{4}}{\delta J^{4}\left(\tau_{1}\right)}\right]\left.K_{J}\right|_{J=0}}. \tag{13}$$

Step 3: express K_J in terms of the Green function

Let us now use the old trick of decomposing the path into the classical path and the deviations from it,

$$x\left(\tau\right)=x_{\text{cl}}\left(\tau\right)+y\left(\tau\right), \tag{14}$$

along with the boundary conditions $x_{\text{cl}}\left(\tau_{i}\right)=x_{\text{cl}}\left(\tau_{f}\right)=y\left(\tau_{i}\right)=y\left(\tau_{f}\right)=0$. The expression in the exponent of K_J (Eq. 10) can then be written as

$$\begin{aligned}\frac{1}{2}m\left(\dot{x}^{2}+\omega^{2}x^{2}\right)+Jx &= \frac{1}{2}m\left(\dot{x}_{\text{cl}}^{2}+\omega^{2}x_{\text{cl}}^{2}\right)+Jx_{\text{cl}}\\ &+ \frac{1}{2}m\left(\dot{y}^{2}+\omega^{2}y^{2}\right)\\ &+ m\left(\dot{x}_{\text{cl}}\dot{y}+\omega^{2}x_{\text{cl}}y\right)+Jy.\end{aligned} \tag{15}$$

The last line in Eq. (15) actually gives zero contribution to the path integral in Eq. (10). In order to prove that, first note that

$$m\left(\dot{x}_{\text{cl}}\dot{y}+\omega^{2}x_{\text{cl}}y\right)+Jy=m\frac{d}{d\tau}\left(\dot{x}_{\text{cl}}y\right)-\left(m\ddot{x}_{\text{cl}}-m\omega^{2}x_{\text{cl}}-J\right)y. \tag{16}$$

Next, consider the Hamiltonian in the exponent of Eq. (10),

$$H=\frac{1}{2}m\left(\left(\frac{dx}{d\tau}\right)^{2}+\omega^{2}x^{2}\right)+Jx=\frac{p^{2}}{2m}+\frac{1}{2}m\omega^{2}x^{2}+Jx. \tag{17}$$

the Hamilton equations determine the equation of motion for the classical path:

$$\frac{dx_{\text{cl}}}{dt}=\frac{\partial H}{\partial p_{\text{cl}}}=\frac{p_{\text{cl}}}{m} \tag{18}$$

$$\frac{dp_{\text{cl}}}{dt}=-\frac{\partial H}{\partial x_{\text{cl}}}=-m\omega^{2}x_{\text{cl}}-J \tag{19}$$

$$\Longrightarrow\ m\frac{d^{2}x_{\text{cl}}}{dt^{2}}+m\omega^{2}x_{\text{cl}}+J=0. \tag{20}$$

But in terms of the imaginary time $\tau=it$ we get

$$m\frac{d^{2}x_{\text{cl}}}{d\tau^{2}}-m\omega^{2}x_{\text{cl}}-J=0. \tag{21}$$

Therefore we see from Eqs. (16) and (21) that the last line of Eq. (15) is just $m\frac{d}{d\tau}\left(\dot{x}_{\text{cl}}y\right)$. Upon integrating this term over τ in Eq. (10) we will get boundary terms at τ_i and τ_f. But these boundary terms vanish because $y\left(\tau_{i}\right)=y\left(\tau_{f}\right)=0$.

Therefore, we conclude that K_J can be expressed as

$$K_{J}\left(0,-i\beta\hbar/2;0,i\beta\hbar/2\right)=\mathrm{e}^{-\frac{1}{\hbar}\int_{-\beta\hbar/2}^{\beta\hbar/2}\left[\frac{1}{2}m\left(\dot{x}_{\text{cl}}^{2}+\omega^{2}x_{\text{cl}}^{2}\right)+Jx_{\text{cl}}\right]d\tau}\int_{y\left(\tau_{i}\right)=0}^{y\left(\tau_{f}\right)=0}\mathcal{D}y\left(\tau\right)\mathrm{e}^{-\frac{1}{\hbar}\int_{-\beta\hbar/2}^{\beta\hbar/2}\frac{1}{2}m\left(\dot{y}^{2}+\omega^{2}y^{2}\right)d\tau}. \tag{22}$$

The path integral can be obtained from the propagator of the standard harmonic oscillator (analytically continued to imaginary times):

$$K_J(0,-i\beta\hbar/2;0,i\beta\hbar/2)=\sqrt{\frac{m\omega}{2\pi\hbar\sinh(\omega\beta\hbar)}}\exp\left\{-\frac{1}{\hbar}\int_{-\beta\hbar/2}^{\beta\hbar/2}\left[\frac{1}{2}m\left(\dot{x}_{\rm cl}^2+\omega^2x_{\rm cl}^2\right)+Jx_{\rm cl}\right]d\tau\right\}. \tag{23}$$

Let us perform integration by parts over the kinetic term:

$$\frac{1}{2}m\int_{-\beta\hbar/2}^{\beta\hbar/2}\dot{x}_{\rm cl}^2d\tau=\frac{1}{2}m\int_{-\beta\hbar/2}^{\beta\hbar/2}\dot{x}_{\rm cl}dx_{\rm cl}=\frac{1}{2}m\left[\dot{x}_{\rm cl}x_{\rm cl}\right]_{-\beta\hbar/2}^{\beta\hbar/2}-\frac{1}{2}m\int_{-\beta\hbar/2}^{\beta\hbar/2}x_{\rm cl}\ddot{x}_{\rm cl}d\tau. \tag{24}$$

Note that the boundary terms vanish since we calculate the propagator for $x_{\rm cl}(\tau_i)=x_{\rm cl}(\tau_f)=0$. Therefore, Eq. (23) becomes

$$K_J(0,-i\beta\hbar/2;0,i\beta\hbar/2)=\sqrt{\frac{m\omega}{2\pi\hbar\sinh(\omega\beta\hbar)}}\exp\left\{\frac{1}{\hbar}\int_{-\beta\hbar/2}^{\beta\hbar/2}\left[\frac{1}{2}x_{\rm cl}\left(m\frac{d^2x_{\rm cl}}{d\tau^2}-m\omega^2x_{\rm cl}\right)-Jx_{\rm cl}\right]d\tau\right\}. \tag{25}$$

Note that according to Eq. (21) the expression in the parenthesis is simply J. Thus

$$K_J(0,-i\beta\hbar/2;0,i\beta\hbar/2)=\sqrt{\frac{m\omega}{2\pi\hbar\sinh(\omega\beta\hbar)}}\exp\left[-\frac{1}{2\hbar}\int_{-\beta\hbar/2}^{\beta\hbar/2}J(\tau)\,x_{\rm cl}(\tau)\,d\tau\right]. \tag{26}$$

The classical path obeys the equation of motion (Eq. 21). The solution can be expressed as

$$x_{\rm cl}(\tau)=x_{\rm hom}(\tau)+\int_{-\beta\hbar/2}^{\beta\hbar/2}G(\tau,\tau')\,J(\tau')\,d\tau', \tag{27}$$

where $x_{\rm hom}(\tau)$ is the homogeneous solution to Eq. (21) when $J=0$ and $G(\tau,\tau')$ is the Green function. The homogeneous solution can be easily obtained:

$$x_{\rm hom}(\tau)=A\mathrm{e}^{\omega\tau}+B\mathrm{e}^{-\omega\tau}. \tag{28}$$

By definition, the Green function vanishes at the end points $\pm\beta\hbar/2$. Because our boundary conditions are $x_{\rm cl}(\pm\beta\hbar/2)=0$, we learn from Eq. (28) that $A=B=0$. Thus, plugging Eq. (27) back in Eq. (26) (with $x_{\rm hom}=0$) results

$$\boxed{K_J(0,-i\beta\hbar/2;0,i\beta\hbar/2)=\sqrt{\frac{m\omega}{2\pi\hbar\sinh(\omega\beta\hbar)}}\exp\left[-\frac{1}{2\hbar}\int_{-\beta\hbar/2}^{\beta\hbar/2}\int_{-\beta\hbar/2}^{\beta\hbar/2}J(\tau)\,G(\tau,\tau')\,J(\tau')\,d\tau d\tau'\right]}. \tag{29}$$

Step 4: solve for the Green function

The Green function satisfies the equation

$$m\left(\frac{d^2}{d\tau^2}-\omega^2\right)G(\tau,\tau')=\delta(\tau-\tau'). \tag{30}$$

For $\tau\neq\tau'$, the solution is a combination of cosh and sinh. Because the Green function vanishes at the end points $\pm\beta\hbar/2$, we can express the solution as

$$G(\tau,\tau')=\begin{cases}A\sinh\left(\omega\tau+\frac{\beta\hbar\omega}{2}\right) & -\frac{\beta\hbar}{2}\leq\tau\leq\tau'\leq\frac{\beta\hbar}{2}\\ B\sinh\left(\omega\tau-\frac{\beta\hbar\omega}{2}\right) & -\frac{\beta\hbar}{2}\leq\tau'\leq\tau\leq\frac{\beta\hbar}{2}\end{cases}. \tag{31}$$

The Green function has to be continuous so by comparing the solutions at $\tau=\tau'$ we get:

$$A\sinh\left(\omega\tau'+\frac{\beta\hbar\omega}{2}\right)=B\sinh\left(\omega\tau'-\frac{\beta\hbar\omega}{2}\right)\equiv C\sinh\left(\omega\tau'+\frac{\beta\hbar\omega}{2}\right)\sinh\left(\omega\tau'-\frac{\beta\hbar\omega}{2}\right). \tag{32}$$

Thus

$$G\left(\tau,\tau'\right)=C\begin{cases}\sinh\left(\omega\tau'-\frac{\beta\hbar\omega}{2}\right)\sinh\left(\omega\tau+\frac{\beta\hbar\omega}{2}\right) & -\frac{\beta\hbar}{2}\leq\tau\leq\tau'\leq\frac{\beta\hbar}{2}\\ \sinh\left(\omega\tau'+\frac{\beta\hbar\omega}{2}\right)\sinh\left(\omega\tau-\frac{\beta\hbar\omega}{2}\right) & -\frac{\beta\hbar}{2}\leq\tau'\leq\tau\leq\frac{\beta\hbar}{2}\end{cases}. \tag{33}$$

In order to find the constant C, we may integrate Eq. (30) around $\tau'\pm\epsilon$. This yields

$$m\left[\left.\frac{dG\left(\tau,\tau'\right)}{d\tau}\right|_{\tau'+\epsilon}-\left.\frac{dG\left(\tau,\tau'\right)}{d\tau}\right|_{\tau'-\epsilon}\right]=1, \tag{34}$$

or

$$m\omega C\left[\sinh\left(\omega\tau'+\frac{\beta\hbar\omega}{2}\right)\cosh\left(\omega\tau'-\frac{\beta\hbar\omega}{2}\right)-\sinh\left(\omega\tau'-\frac{\beta\hbar\omega}{2}\right)\cosh\left(\omega\tau'+\frac{\beta\hbar\omega}{2}\right)\right]=1. \tag{35}$$

We can now use the identity $\sinh\left(a-b\right)=\sinh\left(a\right)\cosh\left(b\right)-\sinh\left(b\right)\cosh\left(a\right)$ to find

$$C=\frac{1}{m\omega\sinh\left(\beta\hbar\omega\right)}. \tag{36}$$

Thus we conclude

$$\boxed{G\left(\tau,\tau'\right)=\frac{1}{m\omega\sinh\left(\beta\hbar\omega\right)}\begin{cases}\sinh\left(\omega\tau'-\frac{\beta\hbar\omega}{2}\right)\sinh\left(\omega\tau+\frac{\beta\hbar\omega}{2}\right) & -\frac{\beta\hbar}{2}\leq\tau\leq\tau'\leq\frac{\beta\hbar}{2}\\ \sinh\left(\omega\tau'+\frac{\beta\hbar\omega}{2}\right)\sinh\left(\omega\tau-\frac{\beta\hbar\omega}{2}\right) & -\frac{\beta\hbar}{2}\leq\tau'\leq\tau\leq\frac{\beta\hbar}{2}\end{cases}}. \tag{37}$$

Step 5: calculate the functional derivatives

As part of our derivation, Eq. (13) tells us that we need to calculate the functional derivatives of K_J. Recall from Eq. (29) that $K_J\propto\exp\left[-\frac{1}{2\hbar}\int\int J\left(\tau\right)G\left(\tau,\tau'\right)J\left(\tau'\right)d\tau d\tau'\right]$. Therefore, the first functional derivative of K_J is

$$\frac{\delta K_J}{\delta J\left(\tau_1\right)}=-\frac{1}{\hbar}\int G\left(\tau,\tau_1\right)J\left(\tau\right)d\tau\times K_J. \tag{38}$$

Ok, so what is this functional derivative and how it exactly works?

The formal definition of the functional derivative of a functional $F\left[x\left(\tau\right)\right]$ with respect to a function $x\left(\tau\right)$ is

$$\frac{\delta F\left[x\left(\tau\right)\right]}{\delta x\left(\tau'\right)}=\lim_{\epsilon\to0}\frac{F\left[x\left(\tau\right)+\epsilon\delta\left(\tau-\tau'\right)\right]-F\left[x\left(\tau\right)\right]}{\epsilon}. \tag{39}$$

I personally don't find that definition very useful. Instead I suggest you to remember that

$$\frac{\delta x\left(\tau\right)}{\delta x\left(\tau'\right)}=\delta\left(\tau-\tau'\right), \tag{40}$$

and all the known rules for ordinary derivatives of composite functions still hold. For example:

$$\frac{\delta}{\delta x\left(\tau'\right)}\int x\left(\tau\right)y\left(\tau\right)d\tau=\int\frac{\delta x\left(\tau\right)}{\delta x\left(\tau'\right)}y\left(\tau\right)d\tau=\int\delta\left(\tau-\tau'\right)y\left(\tau\right)d\tau=y\left(\tau'\right). \tag{41}$$

Notice that the functional derivative acts only on explicit appearances of $x(\tau)$. In principle, it could be that $y(\tau) = x(\tau)$ and so you would argue why don't we also apply the derivative on $y(\tau)$. The answer for this is because the functional derivative acts only on *functionals* of $x(\tau)$. $y(\tau)$ is not a functional — it is simply a function.

Let's look at another example:

$$\begin{aligned}\frac{\delta}{\delta x(\tau')}\exp\left[\int x(\tau)\,y(\tau)\,d\tau\right] &= \exp\left[\int x(\tau)\,y(\tau)\,d\tau\right]\frac{\delta}{\delta x(\tau')}\int x(\tau)\,y(\tau)\,d\tau \\ &= \exp\left[\int x(\tau)\,y(\tau)\,d\tau\right]y(\tau').\end{aligned} \tag{42}$$

Now try to explain to yourself why Eq. (38) is right.

The second functional derivative of K_J is

$$\frac{\delta^2 K_J}{\delta J^2(\tau_1)} = \left[-\frac{1}{\hbar}G(\tau_1,\tau_1) + \frac{1}{\hbar^2}\int\int G(\tau,\tau_1)\,G(\tau',\tau_1)\,J(\tau)\,J(\tau')\,d\tau d\tau'\right]\times K_J. \tag{43}$$

The third functional derivative of K_J is

$$\begin{aligned}\frac{\delta^3 K_J}{\delta J^3(\tau_1)} &= \left[\frac{3}{\hbar^2}G(\tau_1,\tau_1)\int G(\tau,\tau_1)\,J(\tau)\,d\tau\right. \\ &\quad\left.-\frac{1}{\hbar^3}\int\int\int G(\tau,\tau_1)\,G(\tau',\tau_1)\,G(\tau'',\tau_1)\,J(\tau)\,J(\tau')\,J(\tau'')\,d\tau d\tau' d\tau''\right]\times K_J.\end{aligned} \tag{44}$$

Thus, finally:

$$\boxed{\left.\frac{\delta^4 K_J}{\delta J^4(\tau_1)}\right|_{J=0} = \frac{3}{\hbar^2}G^2(\tau_1,\tau_1)\,K_{J=0}}. \tag{45}$$

Step 6: plug everything

Combining Eqs. (13) and (45), we get:

$$K(0,-i\beta\hbar/2;0,i\beta\hbar/2) = \left[1 - 3\hbar b\int_{-\beta\hbar/2}^{\beta\hbar/2}G^2(\tau,\tau)\,d\tau\right]K_{J=0}. \tag{46}$$

Combining this with Eqs. (5) and (29) gives

$$\begin{aligned}E_0 &= -\lim_{\beta\to\infty}\frac{1}{\beta}\ln\left\{\frac{1}{\sqrt{2\pi\sinh(\omega\beta\hbar)}}\left[1 - 3\hbar b\int_{-\beta\hbar/2}^{\beta\hbar/2}G^2(\tau,\tau)\,d\tau\right]\right\} \\ &= \frac{1}{2}\hbar\omega - \lim_{\beta\to\infty}\frac{1}{\beta}\ln\left[1 - 3\hbar b\int_{-\beta\hbar/2}^{\beta\hbar/2}G^2(\tau,\tau)\,d\tau\right].\end{aligned} \tag{47}$$

The first term is of course the energy of the ground state of the harmonic oscillator. The second term therefore is the energy correction. By approximating $\ln(1-x) \approx -x$, we get

$$\boxed{\Delta E = \lim_{\beta\to\infty}\frac{3\hbar b}{\beta}\int_{-\beta\hbar/2}^{\beta\hbar/2}G^2(\tau,\tau)\,d\tau = \frac{3\hbar^2 b}{4m^2\omega^2}}. \tag{48}$$

Here's a more detailed calculation of Eq. (48). We use Eq. (37) to write

$$\begin{aligned}
\Delta E &= \lim_{\beta\to\infty}\frac{3\hbar b}{\beta}\int_{-\beta\hbar/2}^{\beta\hbar/2}G^2(\tau,\tau)\,d\tau \\
&= \lim_{\beta\to\infty}\frac{3\hbar b}{\beta m^2\omega^2\sinh^2(\beta\hbar\omega)}\int_{-\beta\hbar/2}^{\beta\hbar/2}\sinh^2\left(\omega\tau-\frac{\beta\hbar\omega}{2}\right)\sinh^2\left(\omega\tau+\frac{\beta\hbar\omega}{2}\right)d\tau \\
&= \frac{3\hbar^2 b}{m^2\omega^2}\lim_{y\to\infty}\frac{1}{y\sinh^2(y)}\int_{-y/2}^{y/2}\sinh^2\left(x-\frac{y}{2}\right)\sinh^2\left(x+\frac{y}{2}\right)dx, \qquad (49)
\end{aligned}$$

where in the last step I defined $x\equiv\omega\tau$, $y\equiv\beta\hbar\omega$. The integral can be computed by using the identity $2\sinh\left(x-\frac{y}{2}\right)\sinh\left(x+\frac{y}{2}\right)=\cosh(2x)-\cosh(y)$.

$$\begin{aligned}
\int_{-y/2}^{y/2}\sinh^2\left(x-\frac{y}{2}\right)\sinh^2\left(x+\frac{y}{2}\right)dx &= \frac{1}{4}\int_{-y/2}^{y/2}\left[\cosh(2x)-\cosh(y)\right]^2dx \\
&= \frac{1}{4}\left[\int_{-y/2}^{y/2}\cosh^2(2x)\,dx-2\cosh(y)\int_{-y/2}^{y/2}\cosh(2x)\,dx+\cosh^2(y)\int_{-y/2}^{y/2}dx\right] \\
&= \frac{1}{4}\left[\frac{\frac{1}{2}\sinh(2y)+y}{2}-\sinh(2y)+y\cosh^2(y)\right] \\
&= \frac{y}{8}-\frac{3}{16}\sinh(2y)+\frac{y}{4}\cosh^2(y). \qquad (50)
\end{aligned}$$

According to Eq. (49), we need to take the RHS of Eq. (50), divide it by $y\sinh^2(y)$, and take y to infinity. The first term vanishes since it is proportional to $1/\sinh^2(y)$. The second term also vanishes since it is proportional to $\coth(y)/y$. The only term left is the third which gives $\frac{1}{4}\coth^2(y)\to\frac{1}{4}$. Therefore, we conclude

$$\Delta E=\frac{3\hbar^2 b}{4m^2\omega^2}. \qquad (51)$$