Quantum Mechanics 3 - Class Exercise 10

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Special Relativity

In special relativity we work with 4-vectors, such as the 4-position,

$$x^\mu \equiv (t, \vec{x})\,. \tag{1}$$

Here μ is an index which can have 4 values: 0 (the time component), and 1,2,3 (the spatial components). The convention is to use latin letters (i, j, k, etc...) to denote a 3-vector objects, while greek letters (μ, ν, ρ, etc..) are used for 4-vector objects. We also have the 4-velocity

$$U^\mu \equiv \left(\gamma, \gamma\vec{\beta}\right), \tag{2}$$

and the 4-momentum

$$p^\mu \equiv mU^\mu = \left(m\gamma, m\gamma\vec{\beta}\right) \equiv (E, \vec{p})\,. \tag{3}$$

Inner products between two vectors in special relativity are preformed via the Minkowski metric[1]

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{4}$$

Thus, for two 4-vectors $A^\mu = \left(A^0, \vec{A}\right)$, $B^\mu = \left(B^0, \vec{B}\right)$, the inner product is

$$A^\mu B^\nu \eta_{\mu\nu} = A^0B^0 - \vec{A}\cdot\vec{B} = A^0B^0 - A^1B^1 - A^2B^2 - A^3B^3. \tag{5}$$

We use $\eta_{\mu\nu}$ ($\eta^{\mu\nu}$) to lower (raise) indices:

$$A_\mu \equiv A^\nu \eta_{\mu\nu} = \left(A^0, -\vec{A}\right). \tag{6}$$

So remember that $A_0 = A^0$ and $A_i = -A^i$. In the literature A^μ is called a *contra-variant vector* while A_μ is called a *co-variant vector*. We can use co-variant vectors to perform inner products:

$$A^\mu B_\mu = A^0B^0 - \vec{A}\cdot\vec{B} = A^0B^0 - A^1B^1 - A^2B^2 - A^3B^3. \tag{7}$$

You can prove easily from the definition of U^μ in Eq. (2) that $U^\mu U_\mu = 1$. As a result, we have from Eq. (3) that $p^\mu p_\mu = m^2$. However, from the definition of the inner product, $p^\mu p_\mu = E^2 - \vec{p}^{\,2}$, thus we obtain the famous relation

$$\boxed{E^2 = m^2 + \vec{p}^{\,2}}. \tag{8}$$

Photons (which are massless) obey $E = |\vec{p}|$.

[1] This is the common convention in the particle physics community. In the general relativity and cosmology community people use $\eta_{\mu\nu} = \mathrm{diag}\,(-1, 1, 1, 1)$. This is very confusing, but that's life.

Classical Fields

According to classical Lagrangian mechanics, any physical system can be described by the set of its generalized coordinates $\{q\}_{i=1}^{N}$ and their derivatives $\{\dot{q}\}_{i=1}^{N}$. The action is then given by integrating the Lagrangian over time,

$$S = \int dt L\left(\{q\}, \{\dot{q}\}, t\right). \tag{9}$$

The equations of motion are derived from the Euler-Lagrange (EL) equations,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}, \qquad i = 1, \ldots, N. \tag{10}$$

In the limit where there is an infinite number of degrees of freedom ($N \to \infty$), the system in now described in terms of fields $\varphi_A\left(x\right)$. Here, $x = x^\mu = (t, \vec{x})$ acts as a label of spacetime (similarly to the role of i in Eq. (10)), while A labels different fields (e.g. different components of the EM field). The Lagrangian is then given by

$$L = \int dt \mathcal{L}\left(\varphi_A, \partial_\mu \varphi_A\right), \tag{11}$$

where $\mathcal{L}$ is the Lagrangian density[2], $\partial_\mu \equiv \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)$, and the action is

$$S = \int d^4x \mathcal{L}. \tag{12}$$

Now, the EL equations are

$$\boxed{\partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \varphi_A\right)} = \frac{\partial \mathcal{L}}{\partial \varphi_A}} \tag{13}$$

Transformations

There are two kinds of transformations we can consider.

1. **Internal transformations**. These are transformations that act only on the fields themselves, but not on the spacetime coordinates.

$$\varphi_A \to \varphi'_A = \varphi_A + \delta^{(0)}\varphi_A. \tag{14}$$

   For example, consider the (infinitesimal) U(1) transformation,

$$\varphi_A \to \varphi'_A = \mathrm{e}^{i\epsilon}\varphi_A \qquad \Longrightarrow \qquad \delta^{(0)}\varphi_A = i\epsilon\varphi_A. \tag{15}$$

2. **External transformations**. These are transformations of the spacetime coordinates. For infinitesimal external transformations we have

$$x^\mu \to x'^\mu = x^\mu + \delta x^\mu. \tag{16}$$

[2] For brevity, from now on each time we say "Lagrangian" we actually refer to $\mathcal{L}$ (rather than L).

Two important examples of external transformations:

(a) **Translations**. These correspond to shifting the coordinates by constants, $\delta x^{\mu}{=}\epsilon^{\mu}$.

(b) **Lorentz transformations**. These correspond to boosts and rotations. In your HW, you will prove that in this case $\delta x^{\mu}{=}\epsilon^{\mu\nu}x_{\nu}$, where $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$ is some anti-symmetric matrix whose values are determined by the (infinitesimal) parameters of the boosts and rotations.

It is important to understand that external transformations affect the fields as well,

$$\delta\varphi_A = (\partial_\mu \varphi_A)\, \delta x^\mu + \delta^{(0)}\varphi_A. \tag{17}$$

Here, while the first piece corresponds to Taylor expansion of the field at $x^\mu + \delta x^\mu$, the second piece corresponds to internal transformation of the field due to external transformation. We can classify the fields according to their internal transformation due to Lorentz transformation. For example:

(a) **Scalar fields** transform as $\delta\phi = (\partial_\mu \phi)\, \delta x^\mu$ under Lorentz transformations, i.e. $\delta^{(0)}\phi = 0$.

(b) **Vector fields** undergo Lorentz transformations exactly as x^μ, i.e. $\delta^{(0)}A^\mu = \epsilon^{\mu\nu}A_\nu$.

Noether theorem

Symmetry transformations are any transformation (either internal or external) that leave the action unchanged,

$$S \to S' = S. \tag{18}$$

According to Noether theorem, for every global[3] symmetry the Lagrangian possesses there is a corresponding conserved charge. The Noether current reads

$$\boxed{j_{\mu a} = \left(\frac{\partial \mathcal{L}}{\partial\left(\partial^\mu \varphi_A\right)}\partial_\nu \varphi_A - \eta_{\mu\nu}\mathcal{L}\right)\frac{\partial \delta x^\nu}{\partial \epsilon^a} + \frac{\partial \mathcal{L}}{\partial\left(\partial^\mu \varphi_A\right)}\frac{\partial\left(\delta^{(0)}\varphi_A\right)}{\partial \epsilon^a}}, \tag{19}$$

where ϵ^a is the infinitesimal parameter of the symmetry transformation (there could be several symmetry transformations, each of which has different ϵ^a, resulting several Noether currents). Note that A in Eq. (19) is a dummy index, thus all the fields in the Lagrangian have to be summed up. Noether current is conserved in the sense that

$$\partial^\mu j_{\mu a} = 0. \tag{20}$$

From here it is straightforward to show (using Gauss theorem) that the Noether charge Q_a defined below is a conserved quantity.

$$Q_a \equiv \int d^3x j_{0a} \qquad \frac{dQ_a}{dt} = 0. \tag{21}$$

[3] Global symmetry means that the transformation parameters do not depend on the spacetime coordinates (as opposed to local symmerty).

Energy and momentum

Every physical theory should to be invariant under translations[4]. Since for translations $\delta x^\nu = \epsilon^\nu$ and $\delta^{(0)}\varphi_A = 0$, the Noether currents in the case of translations correspond to the energy-momentum tensor,

$$T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial\left(\partial^\mu \varphi_A\right)}\partial_\nu \varphi_A - \eta_{\mu\nu}\mathcal{L}. \tag{22}$$

and the Noether charges are the components of the 4-momentum,

$$p^\mu = \int d^3x T^{0\mu}. \tag{23}$$

Specifically,

$$\boxed{H = E = \int d^3x T^{00}}. \tag{24}$$

$$\boxed{p^i = \int d^3x T^{0i}}. \tag{25}$$

Angular momentum

We can also expect that physical theories are Lorentz invariant[5]. For Lorentz transformations we have $\delta x^\mu = \epsilon^{\mu\nu}x_\nu$. Because $\epsilon^{\mu\nu}$ is anti-symmetric, the Noether currents in this case are given by

$$J_{\mu\alpha\beta} = \left(T_{\mu\alpha}x_\beta - T_{\mu\beta}x_\alpha\right) + \frac{\partial \mathcal{L}}{\partial\left(\partial^\mu \varphi_A\right)}\frac{\partial\left(\delta^{(0)}\varphi_A\right)}{\partial \epsilon^{\alpha\beta}}, \tag{26}$$

and the Noether charges are the generators of the Lorentz group

$$M_{\mu\nu} = \int d^3x J_{0\mu\nu}. \tag{27}$$

The relation between these conserved quantities and the components of the angular momentum is given by

$$\boxed{J^i = \frac{1}{2}\epsilon^{ijk}M^{jk} = L^i + S^i}, \tag{28}$$

where in the last equality we identified L^i (S^i) with the first (second) term of Eq. (26). Therefore, we see that scalar fields (which have $\delta^{(0)}\varphi_A = 0$ under Lorentz transformations) have zero spin. This is why spin-0 bosons are always described with scalar fields. On the other hand, it can be shown that spin-1 bosons correspond to vector fields. Finally, spin-1/2 fermions correspond to Dirac fields (which are beyond the scope of this course).

[4]Mathematically, this statement implies that physical Lagrangians depend only on the fields and their derivatives, but not on x^μ.

[5]This statement implies that $\mathcal{L}$ must be a Lorentz scalar.

Question 1

The Lagrangian of the complex scalar field is given by[6]

$$\mathcal{L} = \partial^\mu \phi^* \partial_\mu \phi - m^2 \phi^* \phi. \tag{29}$$

1. Derive the EL equations for the complex scalar field.
2. Find the momentum densities conjugate to the fields.
3. Solve the EL equations.
4. Find the Hamiltonian and the momentum for this theory.
5. Does this Lagrangian possess any internal symmetry? If so, find the corresponding Noether current and charge.

Solution to 1

Here in this problem we have two independent fields, $\mathrm{Re}\{\phi\}$ and $\mathrm{Im}\{\phi\}$. Therefore, we need to write the Lagrangian in terms of these two independent fields and then derive the EL equations with respect to $\mathrm{Re}\{\phi\}$ and $\mathrm{Im}\{\phi\}$. However, it can be shown quite easily that the resulting equations are identical to the EL equations if we treat ϕ and ϕ^* as the independent fields. Therefore, we have

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi^*, \qquad \frac{\partial \mathcal{L}}{\partial \phi^*} = -m^2 \phi \tag{30}$$

$$\partial^\mu \frac{\partial \mathcal{L}}{\partial \left(\partial^\mu \phi\right)} = \partial^\mu \partial_\mu \phi^* = \Box \phi^*, \qquad \partial^\mu \frac{\partial \mathcal{L}}{\partial \left(\partial^\mu \phi^*\right)} = \partial^\mu \partial_\mu \phi = \Box \phi, \tag{31}$$

$$\implies \boxed{\left(\Box + m^2\right) \phi\left(x\right) = 0, \qquad \left(\Box + m^2\right) \phi^*\left(x\right) = 0}. \tag{32}$$

The equation for $\phi\left(x\right)$ (or $\phi^*\left(x\right)$) is known as the *Klein-Gordon equation*.

Solution to 2

The momentum densities conjugate to $\phi\left(x\right)$ and $\phi^*\left(x\right)$ are given by

$$\pi_\phi\left(x\right) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}^*\left(x\right) \equiv \pi\left(x\right) \tag{33}$$

$$\pi_{\phi^*}\left(x\right) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^*} = \dot{\phi}\left(x\right) \equiv \pi^*\left(x\right) \tag{34}$$

[6] In field theory, it is customary to work in units where $\hbar = c = 1$. We shall adopt this convention.

Solution to 3

To solve the Klein-Gordon equation, we guess

$$\phi(x) \propto \exp\left(\pm i p_\mu x^\mu\right) = \mathrm{e}^{\pm i p\cdot x}, \tag{35}$$

so in our notation $p\cdot x = p_\mu x^\mu$. By plugging this guess into the Klein-Gordon equation we find a condition for the vector p^μ,

$$p^\mu p_\mu = \left(p^0\right)^2 - \vec{p}^{\,2} = m^2 \implies E_p^2 = m^2 + \vec{p}^{\,2}, \tag{36}$$

where I defined $E_p \equiv p^0$. The solution for $\phi(x)$ is therefore a spectrum of plane-waves as in Eq. (35) that obey the dispersion relation of Eq. (36),

$$\boxed{\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_{\vec{p}}\, \mathrm{e}^{-ip\cdot x} + b^*_{\vec{p}}\, \mathrm{e}^{+ip\cdot x}\right)}. \tag{37}$$

Note that the factors of $(2\pi)^3$ and $\sqrt{2E_p}$ are just conventions, and they could in principle be absorbed in $a_{\vec{p}}$ and $b^*_{\vec{p}}$. Also note that if $a_{\vec{p}} = b_{\vec{p}}$, $\phi(x)$ is real.

From Eq. (33) we find

$$\boxed{\pi(x) = -i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{E_p}{2}} \left(b_{\vec{p}}\, \mathrm{e}^{-ip\cdot x} - a^*_{\vec{p}}\, \mathrm{e}^{+ip\cdot x}\right)}. \tag{38}$$

The coefficients $a_{\vec{p}}$ and $b^*_{\vec{p}}$ can be found from the initial conditions of $\phi(x)$ and $\pi(x)$ via (convince yourself!)

$$a_{\vec{p}} = \int d^3x \frac{\mathrm{e}^{-i\vec{p}\cdot\vec{x}}}{\sqrt{2E_p}} \left[E_p \phi(t=0,\vec{x}) + i\pi^*(t=0,\vec{x})\right] \tag{39}$$

$$b^*_{\vec{p}} = \int d^3x \frac{\mathrm{e}^{+i\vec{p}\cdot\vec{x}}}{\sqrt{2E_p}} \left[E_p \phi(t=0,\vec{x}) - i\pi^*(t=0,\vec{x})\right]. \tag{40}$$

Solution to 4

For the Hamiltonian, we first need to compute the 00 component of the energy-momentum tensor,

$$T^{00} = \frac{\partial \mathcal{L}}{\partial(\partial_0\phi)}\partial^0\phi + \frac{\partial \mathcal{L}}{\partial(\partial_0\phi^*)}\partial^0\phi^* - \eta^{00}\mathcal{L} = \dot{\phi}^*\dot{\phi} + \dot{\phi}\dot{\phi}^* - \partial^\mu\phi^*\partial_\mu\phi + m^2\phi^*\phi \tag{41}$$

$$= 2\dot{\phi}^*\dot{\phi} - \left(\dot{\phi}^*\dot{\phi} - \vec{\nabla}\phi\cdot\vec{\nabla}\phi^*\right) + m^2\phi^*\phi = \left|\dot{\phi}\right|^2 + \left|\vec{\nabla}\phi\right|^2 + m^2|\phi|^2, \tag{42}$$

and thus

$$\boxed{H = \int d^3x T^{00} = \int d^3x \left[\left|\dot{\phi}\right|^2 + \left|\vec{\nabla}\phi\right|^2 + m^2|\phi|^2\right]}. \tag{43}$$

For the momentum, we need the $0i$ component of the energy-momentum tensor,

$$T^{0i} = \frac{\partial \mathcal{L}}{\partial(\partial_0\phi)}\partial^i\phi + \frac{\partial \mathcal{L}}{\partial(\partial_0\phi^*)}\partial^i\phi^* - \eta^{0i}\mathcal{L} = -\pi\partial_i\phi - \pi^*\partial_i\phi^*, \tag{44}$$

where the minus sign at second equality is due to $\partial^i = -\partial_i$. Therefore

$$\boxed{\vec{p} = -\int d^3x \left[\pi(x)\vec{\nabla}\phi(x) + \pi^*(x)\vec{\nabla}\phi^*(x)\right]}. \tag{45}$$

Solution to 5

The Lagrangian has U(1) symmetry,

$$\phi \to \phi' = \mathrm{e}^{i\alpha}\phi \qquad \phi^* \to \phi'^* = \mathrm{e}^{-i\alpha}\phi^*, \tag{46}$$

for any real value of α. For an infinitesimal U(1) transformation, we have $\delta^{(0)}\phi = i\epsilon\phi$, $\delta^{(0)}\phi^* = -i\epsilon\phi^*$, $\delta x^\mu = 0$, and therefore the Noether current is

$$j_\mu = \frac{\partial \mathcal{L}}{\partial\left(\partial^\mu \phi\right)}\frac{\partial\left(\delta^{(0)}\phi\right)}{\partial\epsilon} + \frac{\partial \mathcal{L}}{\partial\left(\partial^\mu \phi^*\right)}\frac{\partial\left(\delta^{(0)}\phi^*\right)}{\partial\epsilon} = i\left(\partial_\mu\phi^*\right)\phi - i\left(\partial_\mu\phi\right)\phi^* = -2\,\mathrm{Im}\{\left(\partial_\mu\phi^*\right)\phi\}, \tag{47}$$

and the Noether charge

$$\boxed{Q = \int d^3x j_0 = -2\int d^3x\, \mathrm{Im}\{\pi\left(x\right)\phi\left(x\right)\}}. \tag{48}$$

Question 2

The Lagrangian of the EM field is given by

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - J_\mu A^\mu, \tag{49}$$

where A^μ is the 4-potential (ϕ is the scalar potential while $\vec{A}$ is the vector potential),

$$A^\mu = \left(\phi, \vec{A}\right), \tag{50}$$

$F^{\mu\nu}$ is the electromagnetic tensor, given by

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{bmatrix} 0 & -E^x & -E^y & -E^z \\ E^x & 0 & -B^z & B^y \\ E^y & B^z & 0 & -B^x \\ E^z & -B^y & B^x & 0 \end{bmatrix}, \tag{51}$$

and J^μ is the 4-current (ρ is the charge density, while $\vec{J}$ is the current density)

$$J^\mu = \left(\rho, \vec{J}\right). \tag{52}$$

1. Derive the EL equations for the EM field.
2. Show that the EL equations correspond to two of Maxwell equations.

Solution to 1

We compute

$$\frac{\partial \mathcal{L}}{\partial A^\nu} = -\frac{\partial\left(J_\mu A^\mu\right)}{\partial A^\nu} = -J_\mu\frac{\partial A^\mu}{\partial A^\nu} = -J_\mu\delta^\mu_\nu = -J_\nu \tag{53}$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial\left(\partial^\mu A^\nu\right)} &= -\frac{1}{4}\frac{\partial\left(F^{\rho\sigma}F_{\rho\sigma}\right)}{\partial\left(\partial^\mu A^\nu\right)} = -\frac{1}{4}\frac{\partial\left(\eta_{\alpha\rho}\eta_{\beta\sigma}F^{\rho\sigma}F^{\alpha\beta}\right)}{\partial\left(\partial^\mu A^\nu\right)} = -\frac{\eta_{\alpha\rho}\eta_{\beta\sigma}}{4}\frac{\partial\left(F^{\rho\sigma}F^{\alpha\beta}\right)}{\partial\left(\partial^\mu A^\nu\right)} \\ &= -\frac{\eta_{\alpha\rho}\eta_{\beta\sigma}}{4}\left[F^{\rho\sigma}\frac{\partial F^{\alpha\beta}}{\partial\left(\partial^\mu A^\nu\right)} + F^{\alpha\beta}\frac{\partial F^{\rho\sigma}}{\partial\left(\partial^\mu A^\nu\right)}\right] = -\frac{1}{2}F_{\rho\sigma}\frac{\partial F^{\rho\sigma}}{\partial\left(\partial^\mu A^\nu\right)} \\ &= -\frac{1}{2}F_{\rho\sigma}\frac{\partial\left(\partial^\rho A^\sigma - \partial^\sigma A^\rho\right)}{\partial\left(\partial^\mu A^\nu\right)} = -\frac{1}{2}F_{\rho\sigma}\left(\delta^\rho_\mu\delta^\sigma_\nu - \delta^\rho_\nu\delta^\sigma_\mu\right) = -\frac{1}{2}\left(F_{\mu\nu} - F_{\nu\mu}\right) = -F_{\mu\nu}, \end{aligned} \tag{54}$$

where the last equality is due to the anti-symmetry of $F_{\mu\nu}$, $F_{\nu\mu} = -F_{\mu\nu}$. Therefore, the EL equations for the EM field are

$$\partial^\mu F_{\mu\nu} = J_\nu, \qquad \Longrightarrow \qquad \boxed{\partial_\mu F^{\mu\nu} = J^\nu}. \tag{55}$$

Solution to 2

We first notice from Eq. (51) that we may write the components pf $F^{\mu\nu}$ as

$$F^{00} = 0, \qquad F^{i0} = E^i, \qquad F^{0i} = -E^i, \qquad F^{ij} = -\epsilon^{ijk} B^k. \tag{56}$$

Now we compute the LHS of Eq. (55) for $\nu = 0$,

$$\partial_\mu F^{\mu 0} = \partial_0 F^{00} + \partial_i F^{i0} = \partial_i E^i = \vec{\nabla} \cdot \vec{E}, \tag{57}$$

and now for $\nu = j$,

$$\partial_\mu F^{\mu j} = \partial_0 F^{0j} + \partial_i F^{ij} = -\partial_0 E^j - \partial_i \epsilon^{ijk} B^k = -\frac{\partial E^j}{\partial t} + \epsilon^{ikj} \partial_i B^k = -\frac{\partial E^j}{\partial t} + \left(\vec{\nabla} \times \vec{B}\right)^j. \tag{58}$$

Since $J^\nu = \left(\rho, \vec{J}\right)$ we conclude

$$\boxed{\vec{\nabla} \cdot \vec{E} = \rho} \tag{59}$$

$$\boxed{\vec{\nabla} \times \vec{B} = \vec{J} + \frac{\partial \vec{E}}{\partial t}}. \tag{60}$$

What about the other two Maxwell equations? These equations arise due to the so called *Bianchi identity*,

$$\epsilon_{\mu\nu\rho\sigma} \partial^\nu F^{\rho\sigma} = 0, \tag{61}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the Levi-Civita symbol in 4D,

$$\epsilon_{\mu\nu\rho\sigma} = \begin{cases} +1 & \mu\nu\rho\sigma \text{ is an even permutation of } (0123) \\ -1 & \mu\nu\rho\sigma \text{ is an odd permutation of } (0123) \\ 0 & \text{otherwise} \end{cases}. \tag{62}$$

For $\mu = 0$, the LHS of Eq. (61) is

$$\begin{aligned} \epsilon_{0\nu\rho\sigma} \partial^\nu F^{\rho\sigma} &= \epsilon_{0ijk} \partial^i F^{jk} = -\epsilon_{ijk} \partial^i \epsilon^{jkl} B^l = \partial_i \epsilon^{ijk} \epsilon^{ljk} B^l = \partial_i \left(\delta^{il}\delta^{jj} - \delta^{ij}\delta^{jl}\right) B^l \\ &= \partial_i \left(3\delta^{il} - \delta^{il}\right) B^l = 2\partial_i B^i = 2\vec{\nabla} \cdot \vec{B} \implies \boxed{\vec{\nabla} \cdot \vec{B} = 0}, \end{aligned} \tag{63}$$

while for $\mu = i$ it is

$$\begin{aligned} \epsilon_{i\nu\rho\sigma} \partial^\nu F^{\rho\sigma} &= \epsilon_{i0jk} \partial^0 F^{jk} + \epsilon_{ij0k} \partial^j F^{0k} + \epsilon_{ijk0} \partial^j F^{k0} \\ &= \epsilon_{0ijk} \left(-\partial^0 F^{jk} + \partial^j F^{0k} - \partial^j F^{k0}\right) = \epsilon^{ijk} \left(\partial^0 \epsilon^{jkl} B^l - \partial^j E^k - \partial^j E^k\right) \\ &= \partial^0 \epsilon^{ijk} \epsilon^{jkl} B^l - 2\epsilon^{ijk} \partial^j E^k = 2\partial^0 \delta^{il} B^l + 2\epsilon^{ijk} \partial_j E^k = 2\left(\partial^0 B^i + \epsilon^{ijk} \partial_j E^k\right) \\ &\implies \boxed{\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}. \end{aligned} \tag{64}$$