Quantum Mechanics 3 - Class Exercise 11

28.12.2021

Question 1

For a real massive scalar field, compute $\langle 0|\hat{\phi}(x)|\vec{k}\rangle$.

Solution

In the previous tutorial we found

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_{\vec{p}}\,\mathrm{e}^{-ip\cdot x} + a^*_{\vec{p}}\,\mathrm{e}^{+ip\cdot x}\right) \tag{1}$$

$$\pi(x) = -i\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{E_p}{2}}\left(a_{\vec{p}}\,\mathrm{e}^{-ip\cdot x} - a^*_{\vec{p}}\,\mathrm{e}^{+ip\cdot x}\right), \tag{2}$$

where the coefficients $a_{\vec{p}}$ can be determined from the initial state of $\phi(x)$ and $\pi(x)$. Upon quantization, we promote ϕ and π to be operators in Hilbert space (so we give them cute little "hats"), and require the following commutation relations[1]

$$\left[\hat{\phi}(x), \hat{\pi}(y)\right]_{x^0=y^0} = i\delta^{(3)}(\vec{x}-\vec{y}). \tag{3}$$

$$\left[\hat{\phi}(x), \hat{\phi}(y)\right]_{x^0=y^0} = \left[\hat{\pi}(x), \hat{\pi}(y)\right]_{x^0=y^0} = 0. \tag{4}$$

Here the subscript $x^0 = y^0$ reminds us that Eqs. (3)-(4) hold only when $\hat{\phi}(x)$ and $\hat{\pi}(y)$ are evaluated at equal times. From the above requirement, it follows that

$$\left[\hat{a}_{\vec{p}}, \hat{a}^\dagger_{\vec{p}'}\right] = (2\pi)^3\,\delta^{(3)}(\vec{p}-\vec{p}'). \tag{5}$$

$$\left[\hat{a}_{\vec{p}}, \hat{a}_{\vec{p}'}\right] = \left[\hat{a}^\dagger_{\vec{p}}, \hat{a}^\dagger_{\vec{p}'}\right] = 0. \tag{6}$$

Here's the proof for Eq. (5). In the previous tutorial we found that

$$a_{\vec{p}} = \int d^3x\frac{\mathrm{e}^{-i\vec{p}\cdot\vec{x}}}{\sqrt{2E_p}}\left[E_p\phi(t=0,\vec{x}) + i\pi(t=0,\vec{x})\right] \tag{7}$$

$$a^*_{\vec{p}} = \int d^3x\frac{\mathrm{e}^{+i\vec{p}\cdot\vec{x}}}{\sqrt{2E_p}}\left[E_p\phi(t=0,\vec{x}) - i\pi(t=0,\vec{x})\right]. \tag{8}$$

[1] Remember that we work in units where $c = \hbar = 1$. When we restore $\hbar$, the RHS of Eq. (3) becomes $i\hbar\delta^{(3)}(\vec{x}-\vec{y})$.

After quantization, we have $a_{\vec{p}} \to \hat{a}_{\vec{p}}$ and $a^*_{\vec{p}} \to \hat{a}^\dagger_{\vec{p}}$. When we evaluate the commutator, we will end up with only two non-vanishing commutators,

$$\begin{aligned}
\left[\hat{a}_{\vec{p}}\,, \hat{a}^\dagger_{\vec{p}'}\right] &= -i\int d^3x \frac{\mathrm{e}^{-i\vec{p}\cdot\vec{x}}}{\sqrt{2E_p}} \int d^3y \frac{\mathrm{e}^{+i\vec{p}'\cdot\vec{y}}}{\sqrt{2E_{p'}}} \left(\left[E_p\hat{\phi}\left(x\right), \hat{\pi}\left(y\right)\right]_{t=0} - \left[\hat{\pi}\left(x\right), E_{p'}\hat{\phi}\left(y\right)\right]_{t=0}\right) \\
&= \frac{E_p + E_{p'}}{\sqrt{2E_p}\sqrt{2E_{p'}}} \int d^3x \mathrm{e}^{-i(\vec{p}-\vec{p}')\cdot\vec{x}} = \frac{E_p + E_{p'}}{2\sqrt{E_pE_{p'}}} \left(2\pi\right)^3 \delta^{(3)}\left(\vec{p} - \vec{p}'\right) \\
&= \left(2\pi\right)^3 \delta^{(3)}\left(\vec{p} - \vec{p}'\right). \qquad (9)
\end{aligned}$$

Due to the commutation relations of Eqs. (3)-(4), we can define the (normalized) vacuum state to be

$$\forall\vec{p} \quad \hat{a}_{\vec{p}}\left|0\right\rangle \equiv 0, \qquad \langle 0|0\rangle \equiv 1, \qquad (10)$$

and every 1-particle state is given by

$$\left|\vec{p}\right\rangle \equiv \sqrt{2E_p}\,\hat{a}^\dagger_{\vec{p}}\left|0\right\rangle. \qquad (11)$$

Why do we need that factor of $\sqrt{2E_p}$ in Eq. (11)? First, note that

$$\langle\vec{p}|\vec{q}\rangle = \sqrt{2E_p}\sqrt{2E_q}\langle 0|\hat{a}_{\vec{p}}\,\hat{a}^\dagger_{\vec{q}}|0\rangle = 2\sqrt{E_pE_q}\langle 0|\left[\hat{a}_{\vec{p}}\,, \hat{a}^\dagger_{\vec{q}}\right]|0\rangle = 2E_p\left(2\pi\right)^3\delta^{(3)}\left(\vec{p} - \vec{q}\right). \qquad (12)$$

Hence, these states are not normalized to 1. However, their norm is Lorentz invariant! This is clearly evident if we consider rotations. What about boosts? Let us consider a boost along the z-axis:

$$p_z \to p'_z = \gamma\left(p_z + \beta E_p\right) \qquad (13)$$

$$E_p \to E'_p = \gamma\left(E_p + \beta p_z\right). \qquad (14)$$

We now use the identity

$$\delta\left(f\left(x\right) - f\left(x_0\right)\right) = \frac{1}{\left|f'\left(x_0\right)\right|}\delta\left(x - x_0\right) \qquad (15)$$

to demonstrate how the Dirac delta transforms under boosts

$$\begin{aligned}
\delta^{(3)}\left(\vec{p} - \vec{q}\right) \to \delta^{(3)}\left(\vec{p}' - \vec{q}'\right) &= \frac{\delta^{(3)}\left(\vec{p} - \vec{q}\right)}{\left|\frac{dp'_z}{dp_z}\right|} \overset{(1)}{=} \frac{\delta^{(3)}\left(\vec{p} - \vec{q}\right)}{\left|\gamma\left(1 + \beta\frac{dE_p}{dp_z}\right)\right|} \overset{(2)}{=} \frac{\delta^{(3)}\left(\vec{p} - \vec{q}\right)}{\left|\frac{\gamma}{E_p}\left(E_p + \beta p_z\right)\right|} \\
&\overset{(3)}{=} \frac{\delta^{(3)}\left(\vec{p} - \vec{q}\right)}{\left|\frac{E'_p}{E_p}\right|}. \qquad (16)
\end{aligned}$$

Here equality (1) is due to Eq. (13), equality (2) is due to $E_p = \sqrt{m^2 + p_x^2 + p_y^2 + p_z^2}$, and equality (3) is due to Eq. (14). Since $E_p, E_{p'} > 0$ we conclude

$$E_{p'}\delta^{(3)}\left(\vec{p}' - \vec{q}'\right) = E_p\delta^{(3)}\left(\vec{p} - \vec{q}\right). \qquad (17)$$

Now we can calculate

$$\begin{aligned}\langle 0|\hat{\phi}(x)|\vec{k}\rangle &= \langle 0|\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(\hat{a}_{\vec{p}}\,\mathrm{e}^{-ip\cdot x}+\hat{a}^{\dagger}_{\vec{p}}\,\mathrm{e}^{+ip\cdot x}\right)\sqrt{2E_k}\,\hat{a}^{\dagger}_{\vec{k}}|0\rangle \\ &= \int\frac{d^3p}{(2\pi)^3}\sqrt{\frac{E_k}{E_p}}\mathrm{e}^{-ip\cdot x}\langle 0|\hat{a}_{\vec{p}}\;\hat{a}^{\dagger}_{\vec{k}}|0\rangle = \int\frac{d^3p}{(2\pi)^3}\sqrt{\frac{E_k}{E_p}}\mathrm{e}^{-ip\cdot x}\langle 0|\left[\hat{a}_{\vec{p}}\,,\hat{a}^{\dagger}_{\vec{k}}\right]|0\rangle \\ &= \int\frac{d^3p}{(2\pi)^3}\sqrt{\frac{E_k}{E_p}}\mathrm{e}^{-ip\cdot x}\,(2\pi)^3\,\delta^{(3)}\left(\vec{p}-\vec{k}\right) = \mathrm{e}^{-ik\cdot x}. \end{aligned} \tag{18}$$

Specifically, at $t=0$ we may write[2]

$$\boxed{\langle 0|\hat{\phi}(\vec{x},t=0)|\vec{k}\rangle = \mathrm{e}^{+i\vec{k}\cdot\vec{x}}}. \tag{19}$$

Note the striking resemblance of that result with the non-relativistic relation of quantum mechanics $\langle\vec{x}\,|\vec{k}\,\rangle \propto \mathrm{e}^{+i\vec{k}\cdot\vec{x}}$. We can therefore interpret the LHS of Eq. (19) as the position-space representation of the single-particle state $|\vec{k}\,\rangle$.

[2] Alternatively, we may move to the Schrödinger picture (in which the operator $\hat{\phi}(\vec{x})$ is time independent) to write $\langle 0|\hat{\phi}(\vec{x})|\vec{k}\rangle = \mathrm{e}^{+i\vec{k}\cdot\vec{x}}$.

Question 2

1. Calculate the propagator $U(t) = \langle \vec{x}+\vec{r}|\mathrm{e}^{-i\hat{H}t}|\vec{x}\rangle$ for the non-relativistic free Hamiltonian, $\hat{H} = \frac{\hat{P}^2}{2m}$, in the limit $|\vec{r}| \gg t,\, m^{-1}$ (i.e. outside the lightcone).
2. Repeat the last item, but now with the relativistic free Hamiltonian, $\hat{H} = \sqrt{m^2 + \hat{P}^2}$.
3. Calculate the amplitude $\langle 0|\hat{\phi}(x)\,\hat{\phi}(y)\,|0\rangle$, where $x-y$ is purely spatial, i.e. $x^0 = y^0$.
4. Calculate the commutator $\left[\hat{\phi}(x)\,,\hat{\phi}(y)\right]$, where $x-y$ is spacelike, i.e. $(x-y)^2 < 0$.

Solution to 1

We calculate

$$\begin{aligned} U(t) &= \langle \vec{x}+\vec{r}|\mathrm{e}^{-i\hat{P}^2 t/2m}|\vec{x}\rangle = \langle \vec{x}+\vec{r}|\mathrm{e}^{-i\hat{P}^2 t/2m}\int d^3p|\vec{p}\rangle\langle\vec{p}|\vec{x}\rangle = \int d^3p\,\mathrm{e}^{-ip^2t/2m}\langle\vec{x}+\vec{r}|\vec{p}\rangle\langle\vec{p}|\vec{x}\rangle \\ &= \frac{1}{(2\pi)^3}\int d^3p\,\mathrm{e}^{-ip^2t/2m}\mathrm{e}^{i\vec{p}\cdot\vec{r}} = \frac{1}{(2\pi)^3}\prod_{n=x,y,z}\int dp^n\,\mathrm{e}^{-i(p^n)^2t/2m+ip^nr^n} \\ &= \frac{1}{(2\pi)^3}\prod_{n=x,y,z}\sqrt{\frac{2\pi m}{it}}\mathrm{e}^{im(r^n)^2/2t} = \left(\frac{m}{2\pi i t}\right)^{3/2}\exp\left(\frac{imr^2}{2t}\right), \end{aligned} \tag{20}$$

where $r \equiv |\vec{r}|$. The amplitude of the propagator does not depend on the separation r, nor on the location of the points with respect to the lightcone. This can be understood via the uncertainty principle, as a free particle with a definite momentum $\vec{p}$ is disperse over all space.

Solution to 2

For the relativistic Hamiltonian we have

$$\begin{aligned} U(t) &= \frac{1}{(2\pi)^3}\int d^3p\,\mathrm{e}^{-it\sqrt{m^2+p^2}}\mathrm{e}^{i\vec{p}\cdot\vec{r}} = \frac{1}{2\pi^2 r}\int_0^\infty dp\,p\sin(pr)\,\mathrm{e}^{-it\sqrt{m^2+p^2}} \\ &= \frac{1}{2\pi^2 r}\frac{irtm^2}{r^2-t^2}K_2\left(m\sqrt{r^2-t^2}\right) \overset{r\gg t}{\approx} \frac{it}{2\pi^2}\frac{m^2}{r^2}K_2(mr) \xrightarrow{mr\gg 1} \frac{itm^4}{(2\pi)^{3/2}}\frac{\mathrm{e}^{-mr}}{(mr)^{5/2}}, \end{aligned} \tag{21}$$

where $K_2(x)$ is the modified Bessel function of the second kind (with $n=2$ index), and I have used the asymptotic expansion for large argument,

$$K_n(x) \xrightarrow{x\gg 1} \sqrt{\frac{\pi}{2x}}\mathrm{e}^{-x}. \tag{22}$$

We see that for the relativistic case, while the propagator outside the lightcone is very small, it still does not vanish, and causality is violated. Perhaps this will be cured with QFT...

Solution to 3

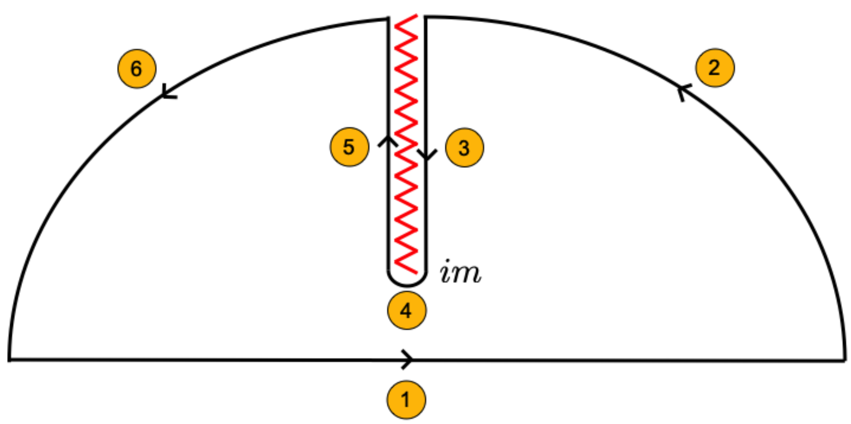
We now need to evaluate

$$\begin{aligned}\langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle &= \langle 0|\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(\hat{a}_{\vec{p}}\,e^{-ip\cdot x}+\hat{a}^{\dagger}_{\vec{p}}\,e^{+ip\cdot x}\right)\int \frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E'_p}}\left(\hat{a}_{\vec{p}'}\,e^{-ip'\cdot y}+\hat{a}^{\dagger}_{\vec{p}'}\,e^{+ip'\cdot y}\right)|0\rangle \\ &= \iint \frac{d^3pd^3p'}{(2\pi)^6}\frac{1}{\sqrt{4E_pE'_p}}e^{-i(p\cdot x-p'\cdot y)}\langle 0|\hat{a}_{\vec{p}}\,\hat{a}^{\dagger}_{\vec{p}'}\,|0\rangle \\ &= \iint \frac{d^3pd^3p'}{(2\pi)^6}\frac{1}{\sqrt{4E_pE'_p}}e^{-i(p\cdot x-p'\cdot y)}\langle 0|\left[\hat{a}_{\vec{p}}\,,\hat{a}^{\dagger}_{\vec{p}'}\,\right]|0\rangle \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip\cdot(x-y)} \equiv D(x-y)\,. \end{aligned} \tag{23}$$

Let us now consider the case where $x-y$ is purely spatial ($x^0-y^0=0$, $\vec{x}-\vec{y}=\vec{r}$).

$$\begin{aligned}\langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{i\vec{p}\cdot\vec{r}} = \frac{1}{(2\pi)^2}\int_0^{\infty} dp\frac{p^2}{2E_p}\frac{\left(e^{ipr}-e^{-ipr}\right)}{ipr} \\ &= \frac{-i}{2(2\pi)^2 r}\underbrace{\int_{-\infty}^{\infty} dp\frac{pe^{ipr}}{\sqrt{m^2+p^2}}}_{I_1}. \end{aligned} \tag{24}$$

In order to evaluate the integral, we shall complete it in the complex plane. The integrand has poles at $p=\pm im$, and so we choose the following contour that bypasses the branch cut (marked with red)



The sum of the integrals must vanish according to Cauchy integral theorem,

$$I_1+I_2+I_3+I_4+I_5+I_6=0. \tag{25}$$

It is easy to verify that $I_2=I_4=I_6=0$. Therefore[3],

$$I_1=-I_3-I_5=-2I_3=-2\int_{i\infty}^{im} dp\frac{pe^{ipr}}{\sqrt{m^2+p^2}} = 2i\int_m^{\infty} d\rho\frac{\rho e^{-\rho r}}{\sqrt{\rho^2-m^2}} = 2imK_1(mr)\,, \tag{26}$$

[3]Note that $I_3=I_5$ due to the branch cut of the square root. If we denote $z\equiv\sqrt{m^2+p^2}=|z|e^{i(\arg z+2\pi)}$ then $\sqrt{z}=\exp\left[\frac{1}{2}\ln(z)\right]=\exp\left[\frac{1}{2}(\ln|z|+i\arg z+2\pi i)\right]=\exp\left[\frac{1}{2}(\ln|z|+i\arg z)\right]e^{i\pi}=-\sqrt{z}$. So it does not matter whether we integrate from im to $i\infty$ or vice versa.

where I changed variables with $\rho \equiv -ip$ and used another integral representation of the modified Bessel function of the second kind. Thus, we conclude that

$$\langle 0|\hat{\phi}(x)\,\hat{\phi}(y)\,|0\rangle = \frac{m}{4\pi^2 r} K_1(mr) \xrightarrow{mr\gg 1} \frac{m^2}{2(2\pi)^{3/2}} \frac{\mathrm{e}^{-mr}}{(mr)^{3/2}}, \tag{27}$$

where I used again Eq. (22). So again, it seems that outside the lightcone the propagation amplitude is exponentially vanishing but still nonzero. However...

Solution to 4

In the previous item we saw that the propagation amplitude does not vanish outside the lightcone, implying the violation of causality. To really discuss causality, however, we should not ask whether particles can propagate over spacelike intervals, but whether a *measurement* performed at one point can affect a measurement at another point whose separation from the first is spacelike.

Recall that in quantum mechanics, we can measure two observables simultaneously only if their Hermitian operators commute. If we have two observables, A and B, and they commute, then we can measure A and then B and the results will be the same as if we measured B and then A. If they don't commute, the results will not be the same: measuring A and then B will produce different results than measuring B and then A. In that case, if we only have access to A and another distant experiment only has access to B, by measuring A several times we can determine whether or not the other experiment has been measuring B or not. Thus, it is crucial that A and B must commute if they are spacelike separated.

Therefore, the true quantity of interest in the context of causality is not the propagation amplitude $D(x-y)$, but rather the commutator $\left[\hat{\phi}(x), \hat{\phi}(y)\right]$. So,

$$\begin{aligned}
\left[\hat{\phi}(x), \hat{\phi}(y)\right] &= \left[\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(\hat{a}_{\vec{p}}\,\mathrm{e}^{-ip\cdot x} + \hat{a}^\dagger_{\vec{p}}\,\mathrm{e}^{+ip\cdot x}\right), \int \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E'_p}} \left(\hat{a}_{\vec{p}'}\,\mathrm{e}^{-ip'\cdot y} + \hat{a}^\dagger_{\vec{p}'}\,\mathrm{e}^{+ip'\cdot y}\right)\right] \\
&= \iint \frac{d^3p d^3p'}{(2\pi)^6} \frac{1}{\sqrt{4E_pE_{p'}}} \left\{\left[\hat{a}_{\vec{p}}, \hat{a}^\dagger_{\vec{p}'}\right] \mathrm{e}^{-i(p\cdot x - p'\cdot y)} + \left[\hat{a}^\dagger_{\vec{p}}, \hat{a}_{\vec{p}'}\right] \mathrm{e}^{-i(p'\cdot y - p\cdot x)}\right\} \\
&= \iint \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left(\mathrm{e}^{-ip\cdot(x-y)} - \mathrm{e}^{-ip\cdot(y-x)}\right) = D(x-y) - D(y-x)\,.
\end{aligned} \tag{28}$$

Let us now focus on the case where $x-y$ is spacelike. Without loss of generality (as we can always rotate the system), we shall assume that x and y are spatially separated only along the z-axis, so

$$x - y = (t \quad 0 \quad 0 \quad z)\,. \tag{29}$$

The spacelike separation of x and y indicates that $z > t$.

We would like to find a different frame of reference where $x-y$ is transformed to $y-x$. This can be done with the following prescription. First, consider a boost along the z-axis that zeros the time component

$$t \to t' = \gamma(t - \beta z) = 0 \quad \text{if} \quad \beta = t/z < 1 \tag{30}$$

$$z \to z' = \gamma(z - \beta t)\,. \tag{31}$$

Note that such a boost exists thanks to $t/z < 1$.

We now rotate the system 180° around the x-axis (or y-axis) to get

$$t' \to t'' = t' = 0 \tag{32}$$

$$z' \to z'' = -z' = \gamma\left(\beta t - z\right). \tag{33}$$

Finally, we boost again, this time in the opposite direction of the z''-axis.

$$t'' \to t''' = \gamma\left(t'' + \beta z''\right) = \gamma^2\left(\beta^2 t - \beta z\right) = \gamma^2\left(\beta^2 - 1\right)t = -t \tag{34}$$

$$z'' \to z''' = \gamma\left(z'' + \beta t''\right) = \gamma^2\left(\beta t - z\right) = \gamma^2\left(\beta^2 - 1\right)z = -z, \tag{35}$$

where I used $\beta = t/z$ and $\gamma = \left(1-\beta^2\right)^{-1/2}$.

Thus, we have found a frame of reference where

$$x - y \to (x-y)''' = y - x. \tag{36}$$

Since $D\left(x-y\right)$ is Lorentz invariant, we conclude that for a spacelike separation $D\left(y-x\right) = D\left(x-y\right)$ and from Eq. (28) we see that

$$\boxed{\left[\hat{\phi}\left(x\right), \hat{\phi}\left(y\right)\right] = 0 \quad \text{if } x-y \text{ is spacelike} \implies \text{causality is maintained in QFT!}} \tag{37}$$

Why $D\left(x-y\right)$ is Lorentz invariant?

From its definition in Eq. (23) it is easy to see that it is invariant under rotations. What about boosts? The exponent $\mathrm{e}^{-ip\cdot x}$ is manifestly Lorentz invariant, so we only need to investigate the transformation properties of d^3p/E_p. Let us consider a boost along the z-axis. The 4-momentum transforms as

$$\begin{bmatrix} E'_p \\ p'_x \\ p'_y \\ p'_z \end{bmatrix} = \begin{bmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} E_p \\ p_x \\ p_y \\ p_z \end{bmatrix} \implies \begin{aligned} E'_p &= \gamma E_p + \gamma\beta p_z \\ p'_x &= p_x \\ p'_y &= p_y \\ p'_z &= \gamma p_z + \gamma\beta E_p \end{aligned} \tag{38}$$

Consider the element d^3p. This element transforms according to

$$\begin{aligned} d^3p \to d^3p' &= dp'_x dp'_y dp'_z = dp_x dp_y \left(\frac{\partial p'_z}{\partial p_z} dp_z + \frac{\partial p'_z}{\partial E_p} dE_p\right) = dp_x dp_y \left(\gamma dp_z + \gamma\beta dE_p\right) \\ &= dp_x dp_y dp_z \left(\gamma + \gamma\beta \frac{\partial E_p}{\partial p_z}\right) = d^3p\left(\gamma + \gamma\beta\frac{\partial E_p}{\partial p_z}\right). \end{aligned} \tag{39}$$

Now, we know that $E_p = \sqrt{m^2 + \vec{p}^{\,2}} = \sqrt{m^2 + p_x^2 + p_y^2 + p_z^2}$, thus:

$$\frac{\partial E_p}{\partial p_z} = \frac{2p_z}{2\sqrt{m^2 + p_x^2 + p_y^2 + p_z^2}} = \frac{p_z}{E_p}. \tag{40}$$

Plugging it back in Eq. (39) we get

$$d^3p' = d^3p\left(\gamma + \gamma\beta\frac{p_z}{E_p}\right) = d^3p\frac{\gamma E_p + \gamma\beta p_z}{E_p} = d^3p\frac{E'_p}{E_p}, \tag{41}$$

where in the last equality we identified E' from Eq. (38). Thus, we finally get

$$\frac{d^3p'}{E'_p} = \frac{d^3p}{E_p}. \tag{42}$$