Quantum Mechanics 3 - Class Exercise 12

18.1.2023

Quantization of the EM Field

We work in the Coulomb gauge where $\vec{\nabla} \cdot \vec{A} = 0$ and $\phi = 0$. In this gauge the $\vec{A}$ field operator can be expressed as

$$\hat{\vec{A}}\left(\vec{r}, t\right) = \int \frac{d^3k}{\sqrt{(2\pi)^3}} \sqrt{\frac{\hbar c}{2k}} \sum_{\lambda=1,2} \left[\mathrm{e}^{i\left(\vec{k}\cdot\vec{r}-\omega_k t\right)} \vec{\epsilon}_{\vec{k},\lambda} \hat{a}_{\vec{k},\lambda} + \mathrm{e}^{-i\left(\vec{k}\cdot\vec{r}-\omega_k t\right)} \vec{\epsilon}^{\,*}_{\vec{k},\lambda} \hat{a}^{\dagger}_{\vec{k},\lambda} \right]. \tag{1}$$

As photons propagate at the speed of light, the dispersion relation is

$$\omega_k = kc. \tag{2}$$

The vectors $\vec{\epsilon}_{k,\lambda}$ are the polarization vectors of the EM field. Along with the wave-vector $\vec{k}$ they form an orthogonal basis[1]

$$\vec{\epsilon}_{\vec{k},\lambda} \cdot \vec{\epsilon}^{\,*}_{\vec{k},\lambda'} = \delta_{\lambda\lambda'}, \qquad \vec{k} \cdot \vec{\epsilon}_{\vec{k},\lambda} = 0, \qquad \sum_{\lambda=1,2} \epsilon^{i}_{\vec{k},\lambda} \epsilon^{*j}_{\vec{k},\lambda} = \delta_{ij} - \frac{k^i k^j}{k^2}. \tag{3}$$

The operators $\hat{a}^{\dagger}_{\vec{k},\lambda}$, $\hat{a}_{\vec{k},\lambda}$ are the creation and annihilation operators of photons with momentum $\hbar\vec{k}$ and polarization λ. They satisfy the commutation relations[2]

$$\left[\hat{a}_{\vec{k},\lambda}, \hat{a}^{\dagger}_{\vec{k}',\lambda'}\right] = \delta_{\lambda\lambda'} \delta^{(3)}\left(\vec{k} - \vec{k}'\right) \tag{4}$$

$$\left[\hat{a}_{\vec{k},\lambda}, \hat{a}_{\vec{k}',\lambda'}\right] = \left[\hat{a}^{\dagger}_{\vec{k},\lambda}, \hat{a}^{\dagger}_{\vec{k}',\lambda'}\right] = 0. \tag{5}$$

In the Coulomb gauge we have

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t}, \qquad \vec{B} = \vec{\nabla} \times \vec{A}. \tag{6}$$

Thus

$$\hat{\vec{E}}\left(\vec{r}, t\right) = i \int \frac{d^3k}{\sqrt{(2\pi)^3}} \sqrt{\frac{\hbar \omega_k}{2}} \sum_{\lambda=1,2} \left[\mathrm{e}^{i\left(\vec{k}\cdot\vec{r}-\omega_k t\right)} \vec{\epsilon}_{\vec{k},\lambda} \hat{a}_{\vec{k},\lambda} - \mathrm{e}^{-i\left(\vec{k}\cdot\vec{r}-\omega_k t\right)} \vec{\epsilon}^{\,*}_{\vec{k},\lambda} \hat{a}^{\dagger}_{\vec{k},\lambda} \right] \tag{7}$$

$$\hat{\vec{B}}\left(\vec{r}, t\right) = i \int \frac{d^3k}{\sqrt{(2\pi)^3}} \sqrt{\frac{\hbar \omega_k}{2}} \sum_{\lambda=1,2} \hat{k} \times \left[\mathrm{e}^{i\left(\vec{k}\cdot\vec{r}-\omega_k t\right)} \vec{\epsilon}_{\vec{k},\lambda} \hat{a}_{\vec{k},\lambda} - \mathrm{e}^{-i\left(\vec{k}\cdot\vec{r}-\omega_k t\right)} \vec{\epsilon}^{\,*}_{\vec{k},\lambda} \hat{a}^{\dagger}_{\vec{k},\lambda} \right]. \tag{8}$$

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[1] Note that the relations of Eq. (3) hold only in the Coulomb gauge!

[2] You may find in the literature slightly different definitions of the commutation relations, specifically you may find an additional factor of $(2\pi)^3$ in the RHS of Eq. (4). This would in return result a factor of $(2\pi)^3$ instead of $\sqrt{(2\pi)^3}$ in the denominator of Eq. (1), (7) and (8). In any case, please note that the exact normalization of the commutation relations of $\hat{a}^{\dagger}_{k,\lambda}$ and $\hat{a}_{k,\lambda}$ do not matter — it is just a choice of normalization! On the other hand, the commutation relations of the operator fields $\vec{A}$, $\vec{E}$, $\vec{B}$ are physical and should always give the same results!

Question 1

Compute $\left[\hat{E}_i\left(\vec{r}_1,t\right),\hat{B}_j\left(\vec{r}_2,t\right)\right]$.

Solution

We shall calculate first the commutator of the expressions in brackets in Eq. (7) and Eq. (8). We use index notation to ease our calculations.

$$\begin{aligned}
&\epsilon_{jnm}\hat{k}'_n\left[\mathrm{e}^{i\left(\vec{k}\cdot\vec{r}_1-\omega_k t\right)}\epsilon^i_{\vec{k},\lambda}\hat{a}_{\vec{k},\lambda}-\mathrm{e}^{-i\left(\vec{k}\cdot\vec{r}_1-\omega_k t\right)}\epsilon^{*i}_{\vec{k},\lambda}\hat{a}^\dagger_{\vec{k},\lambda},\mathrm{e}^{i\left(\vec{k}'\cdot\vec{r}_2-\omega_{k'}t\right)}\epsilon^m_{\vec{k}',\lambda'}\hat{a}_{\vec{k}',\lambda'}-\mathrm{e}^{-i\left(\vec{k}'\cdot\vec{r}_2-\omega_{k'}t\right)}\epsilon^{*m}_{\vec{k}',\lambda'}\hat{a}^\dagger_{\vec{k}',\lambda'}\right]\\
&=-\epsilon_{jnm}\hat{k}'_n\left\{\mathrm{e}^{-i\left(\vec{k}'\cdot\vec{r}_2-\vec{k}\cdot\vec{r}_1+(\omega_k-\omega_{k'})t\right)}\epsilon^i_{\vec{k},\lambda}\epsilon^{*m}_{\vec{k}',\lambda'}\left[\hat{a}_{\vec{k},\lambda},\hat{a}^\dagger_{\vec{k}',\lambda'}\right]+\mathrm{e}^{i\left(\vec{k}'\cdot\vec{r}_2-\vec{k}\cdot\vec{r}_1+(\omega_k-\omega_{k'})t\right)}\epsilon^{*i}_{\vec{k},\lambda}\epsilon^m_{\vec{k}',\lambda'}\left[\hat{a}^\dagger_{\vec{k},\lambda},\hat{a}_{\vec{k}',\lambda'}\right]\right\}\\
&=\epsilon_{jnm}\hat{k}'_n\delta^{(3)}\left(\vec{k}-\vec{k}'\right)\delta_{\lambda\lambda'}\left\{\mathrm{e}^{i\left(\vec{k}'\cdot\vec{r}_2-\vec{k}\cdot\vec{r}_1+(\omega_k-\omega_{k'})t\right)}\epsilon^{*i}_{\vec{k},\lambda}\epsilon^m_{\vec{k}',\lambda'}-\mathrm{e}^{-i\left(\vec{k}'\cdot\vec{r}_2-\vec{k}\cdot\vec{r}_1+(\omega_k-\omega_{k'})t\right)}\epsilon^i_{\vec{k},\lambda}\epsilon^{*m}_{\vec{k}',\lambda'}\right\}\\
&=\epsilon_{jnm}\hat{k}_n\delta^{(3)}\left(\vec{k}-\vec{k}'\right)\delta_{\lambda\lambda'}\left(\mathrm{e}^{i\vec{k}\cdot\Delta\vec{r}}\epsilon^{*i}_{\vec{k},\lambda}\epsilon^m_{\vec{k},\lambda}-\mathrm{e}^{-i\vec{k}\cdot\Delta\vec{r}}\epsilon^i_{\vec{k},\lambda}\epsilon^{*m}_{\vec{k},\lambda}\right),
\end{aligned}\tag{9}$$

where in the last step I defined $\Delta\vec{r}\equiv\vec{r}_2-\vec{r}_1$. We need to apply on that expression $\int d^3k\int d^3k'\sum_\lambda\sum_{\lambda'}$. The $\int d^3k'$ and $\sum_{\lambda'}$ the will make the $\delta^{(3)}\left(\vec{k}-\vec{k}'\right)\delta_{\lambda\lambda'}$ go away. Let's apply $\sum_\lambda$ on the remaining expression.

$$\begin{aligned}
&\sum_{\lambda=1,2}\epsilon_{jnm}\hat{k}_n\left(\mathrm{e}^{i\vec{k}\cdot\Delta\vec{r}}\epsilon^{*i}_{\vec{k},\lambda}\epsilon^m_{\vec{k},\lambda}-\mathrm{e}^{-i\vec{k}\cdot\Delta\vec{r}}\epsilon^i_{\vec{k},\lambda}\epsilon^{*m}_{\vec{k},\lambda}\right)=\epsilon_{jnm}\hat{k}_n\left(\mathrm{e}^{i\vec{k}\cdot\Delta\vec{r}}\sum_{\lambda=1,2}\epsilon^{*i}_{\vec{k},\lambda}\epsilon^m_{\vec{k},\lambda}-\mathrm{e}^{-i\vec{k}\cdot\Delta\vec{r}}\sum_{\lambda=1,2}\epsilon^i_{\vec{k},\lambda}\epsilon^{*m}_{\vec{k},\lambda}\right)\\
&=\epsilon_{jnm}\hat{k}_n\left(\mathrm{e}^{i\vec{k}\cdot\Delta\vec{r}}-\mathrm{e}^{-i\vec{k}\cdot\Delta\vec{r}}\right)\left(\delta_{im}-\hat{k}_i\hat{k}_m\right)=2i\sin\left(\vec{k}\cdot\Delta\vec{r}\right)\left(\epsilon_{ijn}\hat{k}_n-\underbrace{\epsilon_{jnm}\hat{k}_m\hat{k}_n}_{=0}\hat{k}_i\right)=2i\sin\left(\vec{k}\cdot\Delta\vec{r}\right)\epsilon_{ijn}\hat{k}_n,
\end{aligned}\tag{10}$$

where I used the orthogonality relation of Eq. (3). $\epsilon_{jnm}\hat{k}_m\hat{k}_n=0$ because it is a summation of a symmetric object ($k_mk_n=k_nk_m$) with an anti-symmetric object ($\epsilon_{jnm}=-\epsilon_{jmn}$). Thus, we conclude

$$\boxed{\left[\hat{E}_i\left(\vec{r}_1,t\right),\hat{B}_j\left(\vec{r}_2,t\right)\right]=i^2\int\frac{d^3k}{(2\pi)^3}\frac{\hbar\omega_k}{2}2i\sin\left(\vec{k}\cdot\Delta\vec{r}\right)\epsilon_{ijn}\hat{k}_n=-i\epsilon_{ijn}\int\frac{d^3k}{(2\pi)^3}\hat{k}_n\hbar\omega_k\sin\left(\vec{k}\cdot\Delta\vec{r}\right)}.\tag{11}$$

This result is very interesting. First, the RHS is time-independent, so this commutator is constant in time. Second, for $i=j$ the RHS vanishes no matter what the value of $\Delta\vec{r}$ is. If $i\neq j$ on the other hand, then for $\Delta\vec{r}\neq 0$ the commutator *does not* vanish[3]. According to the uncertainty principle, this means that for $\vec{r}_1\neq\vec{r}_2$ we *cannot* measure precisely both $E_x\left(\vec{r}_1,t\right)$ and $B_y\left(\vec{r}_2,t\right)$ since

$$\Delta E_x\left(\vec{r}_1,t\right)\cdot\Delta B_y\left(\vec{r}_2,t\right)\geq\frac{1}{2}\left|\left[\hat{E}_x\left(\vec{r}_1,t\right),\hat{B}_y\left(\vec{r}_2,t\right)\right]\right|\neq 0.\tag{12}$$

The conclusion therefore is that $\Delta\vec{E},\Delta\vec{B}\neq 0$ — the EM fields have non-zero fluctuations! This is also true for the vacuum state where $\langle\vec{E}\rangle=\langle\vec{B}\rangle=0$ (convince yourself!). Is this effect can be measurable? Yes! See the Casimir effect...

[3] Although $\sin\left(\vec{k}\cdot\Delta\vec{r}\right)$ is an anti-symmetric function of $\vec{k}$, so too is $\hat{k}_n$, and so the integrand is a symmetric function of $\vec{k}$.

> *Note:* in the classical limit $\Delta\vec{E} \ll \left|\langle\vec{E}\rangle\right|$, $\Delta\vec{B} \ll \left|\langle\vec{B}\rangle\right|$. These conditions are satisfied for states with many photons. So in normal environments where there is an astonishing amount of photons, the EM fluctuations are hardly noticeable.

Question 2 - The Casimir Effect

Consider two conducting parallel plates of size L^2 located at a distance a in vacuum. Calculate the force between the plates due to vacuum fluctuations.

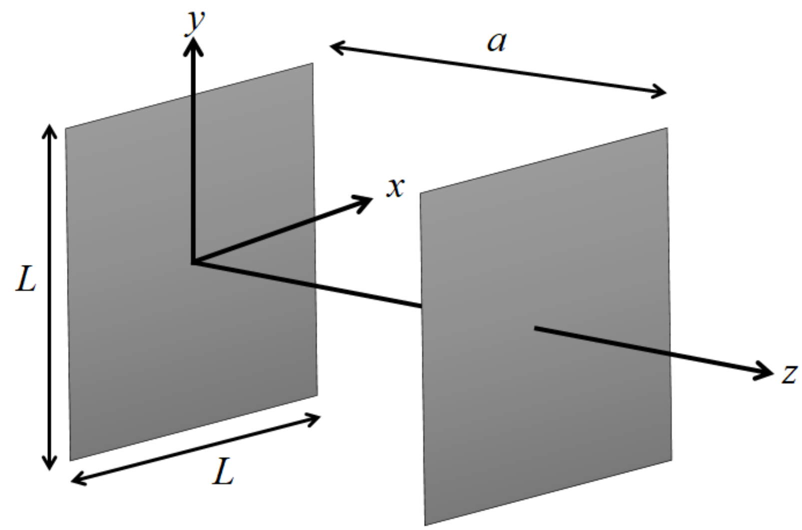

Figure 1: Casimir effect between two conducting parallel plates.

Solution

In class you saw the following expression for the Hamiltonian of the EM field,

$$\hat{H} = \int d^3k \sum_{\lambda=1,2} \frac{\hbar\omega_k}{2} \left(\hat{a}^\dagger_{\vec{k},\lambda}\hat{a}_{\vec{k},\lambda} + \hat{a}_{\vec{k},\lambda}\hat{a}^\dagger_{\vec{k},\lambda}\right). \tag{13}$$

The vacuum expectation energy is therefore

$$\begin{aligned} E_\infty &= \langle 0|\hat{H}|0\rangle = \int d^3k \frac{\hbar\omega_k}{2} \sum_{\lambda=1,2} \langle 0|\hat{a}_{\vec{k},\lambda}\hat{a}^\dagger_{\vec{k},\lambda}|0\rangle = \int d^3k \frac{\hbar\omega_k}{2} \sum_{\lambda=1,2} \langle 0| \left[\hat{a}_{\vec{k},\lambda}, \hat{a}^\dagger_{\vec{k},\lambda}\right] |0\rangle \\ &= \int d^3k \sum_{\lambda=1,2} \frac{\hbar\omega_k}{2} \delta^{(3)}(0) = \int d^3k \sum_{\lambda=1,2} \frac{\hbar\omega_k}{2} \frac{V}{(2\pi)^3} \underset{\substack{V=aL^2 \\ \omega_k=ck}}{=} \frac{\hbar c}{2} \int \frac{L^2 d^2k_\parallel}{(2\pi)^2} \int \frac{a dk_z}{2\pi} \sum_{\lambda=1,2} \sqrt{k_\parallel^2 + k_z^2} \\ &= \frac{\hbar c}{2} \int \frac{L^2 d^2k_\parallel}{(2\pi)^2} \int \frac{a dk_z}{2\pi} \times 2\sqrt{k_\parallel^2 + k_z^2}. \end{aligned} \tag{14}$$

Where does the identity $\delta^{(3)}(0) = V/(2\pi)^3$ come from? Recall that

$$\delta^{(3)}\left(\vec{k}\right) = \frac{1}{(2\pi)^3}\int d^3x e^{i\vec{k}\cdot\vec{x}}. \tag{15}$$

Thus, for $\vec{k}=0$,

$$\delta^{(3)}(0) = \frac{1}{(2\pi)^3}\int d^3x = \frac{V}{(2\pi)^3}. \tag{16}$$

The integral of Eq. (14) is the vacuum expectation energy for a volume of $L^2 \times a$ in the *absence* of the conducting plates, which is equivalent to the plates being infinitely far away from each other (hence the notation E_∞). Let us now calculate the corresponding vacuum expectation energy in the *presence* of the plates (i.e. the plates are at a distance a apart). The calculation is very similar to what was done in Eq. (14), but with two differences:

1. The tangential electric field has to be zero at the plates. This means that k_z now can only have discrete values: $k_z = \frac{n\pi}{a}$, where $n \in \mathbb{N}$.
2. For $n \neq 0$ there are two polarizations that contribute as before. However, there is only one valid polarization for $n = 0$ (the one for which the tangential electric field is zero, $\vec{E} \parallel \hat{z}$).

Thus

$$E_a = \frac{\hbar c}{2}\int \frac{L^2 d^2 k_\parallel}{(2\pi)^2}\left[k_\parallel + 2\sum_{n=1}^{\infty}\sqrt{k_\parallel^2 + \left(\frac{\pi n}{a}\right)^2}\right]. \tag{17}$$

Note that the integrals of both E_∞ and E_a diverge. However, their difference does not (as we prove shortly)

$$\begin{aligned} U &= E_a - E_\infty = \frac{\hbar c}{2}\int \frac{L^2 d^2 k_\parallel}{(2\pi)^2}\left[k_\parallel + 2\sum_{n=1}^{\infty}\sqrt{k_\parallel^2 + \left(\frac{\pi n}{a}\right)^2} - \int_{-\infty}^{\infty}\frac{a dk_z}{2\pi} \times 2\sqrt{k_\parallel^2 + k_z^2}\right] \\ &= \frac{\hbar c}{2}\int \frac{L^2 d^2 k_\parallel}{(2\pi)^2}\left[k_\parallel + 2\sum_{n=1}^{\infty}\sqrt{k_\parallel^2 + \left(\frac{\pi n}{a}\right)^2} - 2\int_0^{\infty} dn\sqrt{k_\parallel^2 + \left(\frac{\pi n}{a}\right)^2}\right]. \end{aligned} \tag{18}$$

The energy difference U is the potential energy due to the presence of the plates alone (without the contribution of the vacuum). Yet, this quantity still diverges. This divergence is due to our approximation of perfect conductors which fails when the wavelengths are shorter than the atoms that build these plates. For such short wavelengths the EM fields are no longer confined between the plates (the photons are capable to breach to the outer region beyond the plates). We would therefore need to apply our calculations only to $k \lesssim k_m$ where k_m^{-1} is of the order of atomic size. This can be done by introducing a cutoff function $f(k)$ which satisfies

$$f(k) = \begin{cases} 1 & k \lesssim k_m \\ 0 & k \gtrsim k_m \end{cases}. \tag{19}$$

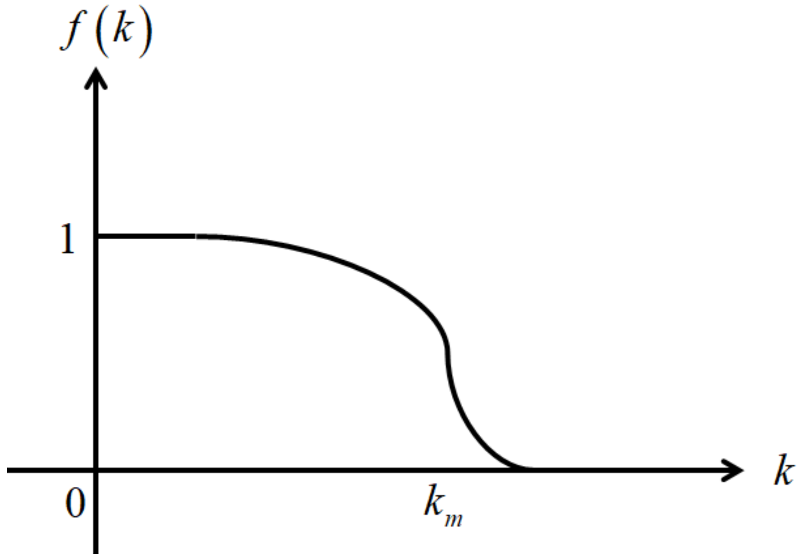

Figure 2: The shape of the cutoff function $f(k)$ which has to be introduced in the calculations of the Casimir effect.

So now, after the integrals include the cutoff function, the expression for U is

$$\begin{aligned} U = \frac{\hbar c L^2}{2} \int_0^\infty \frac{dk_\parallel k_\parallel}{2\pi} \Bigg[& k_\parallel f\left(k_\parallel\right) + 2\sum_{n=1}^{\infty} \sqrt{k_\parallel^2 + \left(\frac{\pi n}{a}\right)^2} f\left(\sqrt{k_\parallel^2 + \left(\frac{\pi n}{a}\right)^2}\right) \\ & - 2\int_0^\infty dn \sqrt{k_\parallel^2 + \left(\frac{\pi n}{a}\right)^2} f\left(\sqrt{k_\parallel^2 + \left(\frac{\pi n}{a}\right)^2}\right)\Bigg] \\ = \frac{\pi^2 \hbar c L^2}{4a^3} \int_0^\infty du \Bigg[& \frac{\sqrt{u}}{2} f\left(\frac{\pi}{a}\sqrt{u}\right) + \sum_{n=1}^{\infty} \sqrt{u+n^2} f\left(\frac{\pi}{a}\sqrt{u+n^2}\right) - \int_0^\infty dn \sqrt{u+n^2} f\left(\frac{\pi}{a}\sqrt{u+n^2}\right)\Bigg], \end{aligned} \tag{20}$$

where in the last step I defined $\sqrt{u} \equiv k_\parallel a/\pi$. Let us now define

$$F(n) \equiv \int_0^\infty du \sqrt{u+n^2} f\left(\frac{\pi}{a}\sqrt{u+n^2}\right) = \int_{n^2}^\infty du \sqrt{u} f\left(\frac{\pi}{a}\sqrt{u}\right). \tag{21}$$

Then we can write Eq. (20) as

$$U = \frac{\pi^2 \hbar c L^2}{4a^3} \left[\frac{1}{2} F(0) + \sum_{n=1}^{\infty} F(n) - \int_0^\infty dn F(n)\right]. \tag{22}$$

In order to calculate the expression in brackets we can use the famous *Euler-Maclaurin formula* that relates the sum of a function with its integral.

$$\sum_{n=1}^{\infty} F(n) = \int_0^\infty dn F(n) + \frac{1}{2}\left[F(\infty) - F(0)\right] + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k!}\left[F^{(2k-1)}(\infty) - F^{(2k-1)}(0)\right], \tag{23}$$

where $F^{(k)}(n)$ is the k'th derivative of $F(n)$ and B_k are the Bernoulli numbers which are given by

$$B_k = \frac{d^k}{dx^k} \frac{x}{e^x - 1}\bigg|_{x=0}. \tag{24}$$

Let us now calculate the first derivative of $F(n)$. It is

$$F'(n) = -2n^2 f\left(\frac{\pi n}{a}\right). \tag{25}$$

Since $f(k \to 0) = 1$, $F'(0) = 0$. Similarly, all the derivatives of $f(k)$ vanish at zero, and the only derivative of $F(n)$ that lasts at zero is the third one,

$$F^{(2k-1)}(0) = -4\delta_{k,2}. \tag{26}$$

From Eq. (21) it should be clear that $F(\infty) = 0$. Notice that if $f(k)$ decays fast enough then all the derivatives of $F(n)$ at infinity vanish[4].

Thus, the Euler-Maclaurin formula simplifies by a lot.

$$\sum_{n=1}^{\infty} F(n) = \int_0^{\infty} dnF(n) - \frac{1}{2}F(0) + 4\frac{B_4}{4!} \tag{27}$$

$$\implies \frac{1}{2}F(0) + \sum_{n=1}^{\infty} F(n) - \int_0^{\infty} dnF(n) = \frac{B_4}{3!} = -\frac{1}{180}, \tag{28}$$

where I used $B_4 = -1/30$. Plugging Eq. (28) back into Eq. (22) yields

$$U = -\frac{\pi^2 \hbar c L^2}{720 a^3}. \tag{29}$$

So the force that acts upon the plates is

$$\boxed{F = -\frac{dU}{da} = -\frac{\pi^2 \hbar c L^2}{240 a^4} = -1.3 \times 10^{-7}\,\mathrm{N}\left(\frac{L}{1\mathrm{cm}}\right)^2 \left(\frac{1\mu\mathrm{m}}{a}\right)^4}. \tag{30}$$

The minus sign tells us that the force between the plates is *attractive*. The force is proportional to a^{-4}, i.e. it grows much stronger for small a. And yet, even for a separation of $a = 1\mu$m between the plates, the force is extremely weak. This force, which is generated only due to the fluctuations of the vacuum is known as the *Casimir force*. It was predicted by Casimir in 1948 and it was first observed only 50 years later in 1998.

Note: notice that our final result of Eq. (30) does not depend on the assumed cutoff wave-number k_m (as it should!). Therefore, k_m can be taken to be arbitrarily large.

[4] Actually, from Eq. (25) we see that $f(k)$ only needs to decay faster than $\mathcal{O}\left(k^{-2}\right)$.