Quantum Mechanics 3 - Class Exercise 13

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Question 1

Calculate the hydrogen transition rate from $|2p\rangle$ to $|1s\rangle$ in the dipole approximation.

Solution

Whenever a hydrogen atom undergoes a transition from the $|2p\rangle$ state ($n = 2$, $\ell = 1$) to the $|1s\rangle$ state ($n = 1$, $\ell = 0$), a photon of energy $E_2 - E_1$ is emitted. In order to calculate the transition rate, we shall apply the Fermi Golden rule,

$$\Gamma_{I\to F} = \frac{2\pi}{\hbar}\left|\langle\psi_F|\hat{V}|\psi_I\rangle\right|^2 \delta\left(E_F - E_I\right). \tag{1}$$

Here,

- $|\psi_I\rangle = |2p\rangle \otimes |0\rangle$ is the initial state, where $|0\rangle$ refers to the ground state of the EM field. The initial energy is $E_I = E_2$.
- $|\psi_F\rangle = |1s\rangle \otimes |\vec{k}, \lambda\rangle$ is the final state, where $|\vec{k}, \lambda\rangle$ refers to the state of a single photon with momentum $\hbar\vec{k}$ and polarization λ. The final energy is $E_F = E_1 + \hbar\omega_k$.

> By the way, since the photon is a spin-1 boson with two polarizations (see next problem), the selection rules of angular momentum imply that the only allowed transitions are those in which $\Delta\ell = 0, \pm 1$ and $\Delta m_z = \pm 1$. For that reason, the transition $2s \to 1s$ is forbidden (unless two photons are emitted).

We will use the dipole approximation, in which

$$\hat{V} = -e\hat{\vec{E}}\left(0\right) \cdot \hat{\vec{r}}. \tag{2}$$

Finally, we need to integrate over all the possible momenta and polarizations of the emitted photon. So now, Eq. (1) becomes

$$\Gamma_{2p\to 1s} = \frac{2\pi e^2}{\hbar}\int d^3k \sum_{\lambda=1,2}\left|\langle\vec{k},\lambda|\hat{E}^i\left(0\right)|0\rangle\langle 1s|\hat{r}^i|2p\rangle\right|^2 \delta\left(\hbar\omega_k - \Delta E\right), \tag{3}$$

where $\Delta E \equiv E_2 - E_1$, and I switched to index notation for the components of $\hat{\vec{r}}$ and $\hat{\vec{E}}$.

We therefore have to evaluate two sandwiches. First, consider $\langle 1s|\hat{r}^i\,|2p\rangle$. This sandwich is not well defined since the $|2p\rangle$ state is degenerate: there are 3 possible values for m_z (0, 1 or -1). We will therefore define the following state,

$$|2p^j\rangle \equiv \begin{cases} \dfrac{|2,1,1\rangle + |2,1,-1\rangle}{\sqrt{2}} & j = x \\ \dfrac{|2,1,1\rangle - |2,1,-1\rangle}{\sqrt{2}} & j = y \\ |2,1,0\rangle & j = z \end{cases}. \tag{4}$$

You can convince yourself that $|2p^j\rangle$ corresponds to the $|2p\rangle$ state that has zero orbital angular momentum at direction j. So we can guess from dimensional considerations

$$\langle 1s|\hat{r}^i|2p^j\rangle = Ca_0\delta^{ij}, \tag{5}$$

where a_0 is the Bohr radius and C is some dimensionless constant, which turns out to be $C = \frac{256}{243\sqrt{2}}$.

Let see that this is really the case for $j = 3$. The wavefunctions of the $|1s\rangle$ and $|2p^3\rangle$ states are given by

$$\psi_{100}\left(\vec{r}\right) = \langle\vec{r}\,|1s\rangle = R_{10}\left(r\right)Y_{00}\left(\theta,\phi\right) = \frac{1}{\sqrt{\pi}a_0^{3/2}}\mathrm{e}^{-r/a_0}. \tag{6}$$

$$\psi_{210}\left(\vec{r}\right) = \langle\vec{r}\,|2p^3\rangle = R_{21}\left(r\right)Y_{10}\left(\theta,\phi\right) = \frac{1}{4\sqrt{2\pi}a_0^{3/2}}\frac{r}{a_0}\mathrm{e}^{-r/2a_0}\cos\theta. \tag{7}$$

Thus

$$\begin{aligned}\langle 1s|\hat{\vec{r}}|2p^3\rangle &= \int d^3r\psi_{100}^*\left(\vec{r}\right)\vec{r}\,\psi_{210}\left(\vec{r}\right) \\ &= \int_{r=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^2\sin\theta drd\theta d\phi\frac{1}{\sqrt{\pi}a_0^{3/2}}\mathrm{e}^{-r/a_0}\begin{pmatrix} r\sin\theta\cos\phi \\ r\sin\theta\sin\phi \\ r\cos\theta\end{pmatrix}\frac{1}{4\sqrt{2\pi}a_0^{3/2}}\frac{r}{a_0}\mathrm{e}^{-r/2a_0}\cos\theta.\end{aligned} \tag{8}$$

Since $\int_{\phi=0}^{2\pi} d\phi\cos\phi = \int_{\phi=0}^{2\pi} d\phi\sin\phi = 0$, the only component for which the integral does not vanish is $j = 3$, so

$$\begin{aligned}\langle 1s|\hat{r}^i|2p^3\rangle &= \delta^{i3}\frac{2\pi}{4\sqrt{2}\pi a_0^4}\int_{r=0}^{\infty} dr\, r^4\mathrm{e}^{-3r/2a_0}\underbrace{\int_{\theta=0}^{\pi} d\theta\sin\theta\cos^2\theta}_{=2/3} = \delta^{i3}\frac{1}{3\sqrt{2}a_0^4}\left(\frac{2a_0}{3}\right)^5\underbrace{\int_{x=0}^{\infty} dx\, x^4\mathrm{e}^{-x}}_{=4!} \\ &= \delta^{i3}\frac{1}{3\sqrt{2}}\left(\frac{2}{3}\right)^5 3\cdot 2^3 a_0 = \delta^{i3}\frac{2^8}{3^5\sqrt{2}}a_0 = \delta^{i3}\frac{256}{243\sqrt{2}}a_0.\end{aligned}$$

Similarly, you may prove that Eq. (5) still holds for $j \neq 3$, but this is not required as we can rotate the system such that $j = 3$ (in other words, $\langle 1s|\hat{r}^i|2p^j\rangle$ is invariant to rotations, as clearly evident from the RHS of Eq. (5)).

Next, the electric field operator is given by

$$\hat{\vec{E}}\left(0\right) = i\int\frac{d^3k}{\sqrt{\left(2\pi\right)^3}}\sqrt{\frac{\hbar\omega_k}{2}}\sum_{\lambda=1,2}\left[\mathrm{e}^{-i\omega_k t}\vec{\epsilon}_{\vec{k},\lambda}\hat{a}_{\vec{k},\lambda} - \mathrm{e}^{+i\omega_k t}\vec{\epsilon}^{\,*}_{\vec{k},\lambda}\hat{a}^\dagger_{\vec{k},\lambda}\right]. \tag{9}$$

so we have

$$\begin{aligned}
\langle\vec{k},\lambda|\hat{E}^{i}\left(0\right)|0\rangle &= \langle 0|\hat{a}_{\vec{k},\lambda}i\int\frac{d^{3}k'}{\sqrt{\left(2\pi\right)^{3}}}\sqrt{\frac{\hbar\omega_{k'}}{2}}\sum_{\lambda'=1,2}\left[\mathrm{e}^{-i\omega'_{k}t}\epsilon^{i}_{\vec{k}',\lambda'}\hat{a}_{\vec{k}',\lambda'}-\mathrm{e}^{+i\omega'_{k}t}\epsilon^{i*}_{\vec{k}',\lambda'}\hat{a}^{\dagger}_{\vec{k}',\lambda'}\right]|0\rangle \\
&= -i\int\frac{d^{3}k'}{\sqrt{\left(2\pi\right)^{3}}}\sqrt{\frac{\hbar\omega_{k'}}{2}}\sum_{\lambda'=1,2}\mathrm{e}^{+i\omega'_{k}t}\epsilon^{i*}_{\vec{k}',\lambda'}\langle 0|\hat{a}_{\vec{k},\lambda}\hat{a}^{\dagger}_{\vec{k}',\lambda'}|0\rangle \\
&= -i\int\frac{d^{3}k'}{\sqrt{\left(2\pi\right)^{3}}}\sqrt{\frac{\hbar\omega_{k'}}{2}}\sum_{\lambda'=1,2}\mathrm{e}^{+i\omega'_{k}t}\epsilon^{i*}_{\vec{k}',\lambda'}\langle 0|\left[\hat{a}_{\vec{k},\lambda},\hat{a}^{\dagger}_{\vec{k}',\lambda'}\right]|0\rangle \\
&= -i\int\frac{d^{3}k'}{\sqrt{\left(2\pi\right)^{3}}}\sqrt{\frac{\hbar\omega_{k'}}{2}}\sum_{\lambda'=1,2}\mathrm{e}^{+i\omega'_{k}t}\epsilon^{i*}_{\vec{k}',\lambda'}\delta_{\lambda\lambda'}\delta^{(3)}\left(\vec{k}-\vec{k}'\right). \\
&= \frac{-i}{\sqrt{\left(2\pi\right)^{3}}}\sqrt{\frac{\hbar\omega_{k}}{2}}\mathrm{e}^{+i\omega_{k}t}\epsilon^{i*}_{\vec{k},\lambda}. \qquad (10)
\end{aligned}$$

Therefore

$$\left|\langle\vec{k},\lambda|\hat{E}^{i}\left(0\right)|0\rangle\langle 1s|\hat{r}^{i}|2p^{j}\rangle\right|^{2}=\frac{C^{2}}{2\left(2\pi\right)^{3}}a_{0}^{2}\hbar\omega_{k}\left|\epsilon^{i*}_{\vec{k},\lambda}\delta^{ij}\right|^{2}=\frac{C^{2}}{2\left(2\pi\right)^{3}}a_{0}^{2}\hbar\omega_{k}\left|\epsilon^{j}_{\vec{k},\lambda}\right|^{2}, \qquad (11)$$

and thus, by plugging Eq. (11) back to Eq. (3), we get

$$\begin{aligned}
\Gamma_{2p^{j}\to 1s} &= \frac{C^{2}e^{2}a_{0}^{2}}{2\left(2\pi\right)^{2}}\int d^{3}k\omega_{k}\delta\left(\hbar\omega_{k}-\Delta E\right)\sum_{\lambda=1,2}\left|\epsilon^{j}_{\vec{k},\lambda}\right|^{2} \\
&= \frac{C^{2}e^{2}a_{0}^{2}}{2\left(2\pi\right)^{2}}c\int dk\,k^{3}\delta\left(\hbar ck-\Delta E\right)\int d\Omega\left(1-\left(\frac{k^{j}}{k}\right)^{2}\right), \qquad (12)
\end{aligned}$$

where the second line follows from $\omega_k = kc$ and $\sum_{\lambda=1,2}\epsilon^{i}_{\vec{k},\lambda}\epsilon^{j*}_{\vec{k},\lambda}=\delta^{ij}-k^{i}k^{j}/k^{2}$. Luckily, the angular integral does not depend on the value of j,

$$\int d\Omega\left(1-\left(\frac{k^{j}}{k}\right)^{2}\right)=\frac{8\pi}{3}\qquad\text{for } j=1,2,3 \qquad (13)$$

which is expected since otherwise the transition rate would depend on the frame of reference. Therefore, once we do the k integral in Eq. (12), we may write

$$\Gamma_{2p\to 1s}=\frac{C^{2}e^{2}a_{0}^{2}}{8\pi^{2}\hbar^{4}c^{3}}\left(\Delta E\right)^{3}\frac{8\pi}{3}=\frac{C^{2}e^{2}a_{0}^{2}}{3\pi\hbar^{4}c^{3}}\left(\Delta E\right)^{3}. \qquad (14)$$

Finally, let us simplify the expression with

$$\Delta E=\frac{3}{4}E_{0}=\frac{3e^{2}}{8a_{0}},\qquad a_{0}=\frac{\hbar}{m_{e}c\alpha},\qquad \alpha\equiv\frac{e^{2}}{4\pi\hbar c}\approx\frac{1}{137}, \qquad (15)$$

yielding

$$\Gamma_{2p\to 1s}=\left(\frac{3}{2}\right)^{8}\alpha^{5}\frac{m_{e}c^{2}}{\hbar}. \qquad (16)$$

To put things into perspective, let us calculate the lifetime of the $|2p\rangle$ state,

$$\boxed{\tau_{2p} = \frac{1}{\Gamma_{2p\to 1s}} = \left(\frac{2}{3}\right)^8 \frac{\alpha^5 \hbar}{m_e c^2} = 1.6\,\text{ns}}, \tag{17}$$

so the transition happens very fast!

Question 2

Show that the photon is a spin-1 particle with $m_z = \pm 1$.

Solution

Recall that the spin of the EM field is given by

$$\vec{S} = \frac{1}{c}\int d^3r \left(\vec{E}\times\vec{A}\right), \tag{18}$$

which is due to the internal transformation of the components of $\vec{A}$ under rotations. Upon quantization, we write[1]

$$\hat{\vec{S}} = \frac{1}{c}\int d^3r \left(\hat{\vec{E}}\times\hat{\vec{A}}\right). \tag{19}$$

Let us now consider a photon that travels along the z direction with some momentum $\vec{k} = k\hat{z}$ and has *linear* polarization λ (the linearity of the polarization will be important in the following derivation). Since $\left[\hat{S}^i, \hat{P}^i\right] = 0$, we expect that this photon is also an eigen-state of $\hat{S}^z$. Therefore, we are interested in finding out what is the z-th component of the spin of the photon, and in particular what are the possible eigen-values of $\hat{S}^z$. So we would like to find out what are the values that m_z can have in

$$\hat{S}^z|\vec{k},\lambda\rangle = \hbar m_z |\vec{k},\lambda\rangle. \tag{20}$$

We shall evaluate

$$\begin{aligned}\hat{S}^z|\vec{k},\lambda\rangle &= \hat{S}^z \hat{a}^\dagger_{\vec{k},\lambda}|0\rangle = \hat{S}^z \hat{a}^\dagger_{\vec{k},\lambda}|0\rangle \overset{(1)}{=} \left[\hat{S}^z, \hat{a}^\dagger_{\vec{k},\lambda}\right]|0\rangle = \left[\frac{1}{c}\int d^3r\left(\hat{E}^x\hat{A}^y - \hat{E}^y\hat{A}^x\right), \hat{a}^\dagger_{\vec{k},\lambda}\right]|0\rangle \\ &= \frac{1}{c}\int d^3r \left[\hat{E}^x\hat{A}^y - \hat{E}^y\hat{A}^x, \hat{a}^\dagger_{\vec{k},\lambda}\right]|0\rangle,\end{aligned} \tag{21}$$

where equality (1) is due to the fact that the vacuum has no angular momentum (and no spin). So our task now is to calculate

$$\left[\hat{E}^x\hat{A}^y, \hat{a}^\dagger_{\vec{k},\lambda}\right] = \left[\hat{E}^x, \hat{a}^\dagger_{\vec{k},\lambda}\right]\hat{A}^y + \hat{E}^x\left[\hat{A}^y, \hat{a}^\dagger_{\vec{k},\lambda}\right] \tag{22}$$

The EM field operators are given by (note that I use here real linear polarization vectors $\vec{\epsilon}_{\vec{k},\lambda}$)

$$\hat{\vec{A}}(\vec{r},t) = \int \frac{d^3k}{\sqrt{(2\pi)^3}}\sqrt{\frac{\hbar c}{2k}}\sum_{\lambda=1,2}\left[e^{i\left(\vec{k}\cdot\vec{r}-\omega_k t\right)}\vec{\epsilon}_{\vec{k},\lambda}\hat{a}_{\vec{k},\lambda} + e^{-i\left(\vec{k}\cdot\vec{r}-\omega_k t\right)}\vec{\epsilon}_{\vec{k},\lambda}\hat{a}^\dagger_{\vec{k},\lambda}\right] \tag{23}$$

$$\hat{\vec{E}}(\vec{r},t) = i\int \frac{d^3k}{\sqrt{(2\pi)^3}}\sqrt{\frac{\hbar\omega_k}{2}}\sum_{\lambda=1,2}\left[e^{i\left(\vec{k}\cdot\vec{r}-\omega_k t\right)}\vec{\epsilon}_{\vec{k},\lambda}\hat{a}_{\vec{k},\lambda} - e^{-i\left(\vec{k}\cdot\vec{r}-\omega_k t\right)}\vec{\epsilon}_{\vec{k},\lambda}\hat{a}^\dagger_{\vec{k},\lambda}\right]. \tag{24}$$

[1] Note that $\hat{\vec{E}}\times\hat{\vec{A}}$ is Hermitian due to the commutation relations of $\hat{\vec{E}}$ and $\hat{\vec{A}}$ so we don't have to write $\hat{\vec{S}} = \frac{1}{2c}\int d^3x\left(\hat{\vec{E}}\times\hat{\vec{A}} - \hat{\vec{A}}\times\hat{\vec{E}}\right)$.

Thus

$$\begin{aligned}
\left[\hat{E}^x, \hat{a}^\dagger_{\vec{k},\lambda}\right] &= \left[i\int \frac{d^3k'}{\sqrt{(2\pi)^3}}\sqrt{\frac{\hbar\omega_{k'}}{2}}\sum_{\lambda'=1,2}\left[\mathrm{e}^{i\left(\vec{k}'\cdot\vec{r}-\omega_{k'}t\right)}\epsilon^x_{\vec{k}',\lambda'}\hat{a}_{\vec{k}',\lambda'} - \mathrm{e}^{-i\left(\vec{k}'\cdot\vec{r}-\omega_{k'}t\right)}\epsilon^x_{\vec{k}',\lambda'}\hat{a}^\dagger_{\vec{k}',\lambda'}\right], \hat{a}^\dagger_{\vec{k},\lambda}\right] \\
&= i\int \frac{d^3k'}{\sqrt{(2\pi)^3}}\sqrt{\frac{\hbar\omega_{k'}}{2}}\sum_{\lambda'=1,2}\mathrm{e}^{i\left(\vec{k}'\cdot\vec{r}-\omega_{k'}t\right)}\epsilon^x_{\vec{k}',\lambda'}\left[\hat{a}_{\vec{k}',\lambda'}, \hat{a}^\dagger_{\vec{k},\lambda}\right] \\
&= i\int \frac{d^3k'}{\sqrt{(2\pi)^3}}\sqrt{\frac{\hbar\omega_{k'}}{2}}\sum_{\lambda'=1,2}\mathrm{e}^{i\left(\vec{k}'\cdot\vec{r}-\omega_{k'}t\right)}\epsilon^x_{\vec{k}',\lambda'}\delta_{\lambda\lambda'}\delta^{(3)}\left(\vec{k}-\vec{k}'\right) \\
&= \frac{i}{\sqrt{(2\pi)^3}}\sqrt{\frac{\hbar\omega_k}{2}}\mathrm{e}^{i\left(\vec{k}\cdot\vec{r}-\omega_k t\right)}\epsilon^x_{\vec{k},\lambda},
\end{aligned} \tag{25}$$

and very similarly

$$\begin{aligned}
\left[\hat{A}^y, \hat{a}^\dagger_{\vec{k},\lambda}\right] &= \left[\int \frac{d^3k'}{\sqrt{(2\pi)^3}}\sqrt{\frac{\hbar c}{2k'}}\sum_{\lambda'=1,2}\left[\mathrm{e}^{i\left(\vec{k}'\cdot\vec{r}-\omega_{k'}t\right)}\epsilon^y_{\vec{k}',\lambda'}\hat{a}_{\vec{k}',\lambda'} + \mathrm{e}^{-i\left(\vec{k}'\cdot\vec{r}-\omega_{k'}t\right)}\epsilon^y_{\vec{k}',\lambda'}\hat{a}^\dagger_{\vec{k}',\lambda'}\right], \hat{a}^\dagger_{\vec{k},\lambda}\right] \\
&= \int \frac{d^3k'}{\sqrt{(2\pi)^3}}\sqrt{\frac{\hbar c}{2k'}}\sum_{\lambda'=1,2}\mathrm{e}^{i\left(\vec{k}'\cdot\vec{r}-\omega_{k'}t\right)}\epsilon^y_{\vec{k}',\lambda'}\left[\hat{a}_{\vec{k}',\lambda'}, \hat{a}^\dagger_{\vec{k},\lambda}\right] \\
&= \int \frac{d^3k'}{\sqrt{(2\pi)^3}}\sqrt{\frac{\hbar c}{2k'}}\sum_{\lambda'=1,2}\mathrm{e}^{i\left(\vec{k}'\cdot\vec{r}-\omega_{k'}t\right)}\epsilon^y_{\vec{k}',\lambda'}\delta_{\lambda\lambda'}\delta^{(3)}\left(\vec{k}-\vec{k}'\right) \\
&= \frac{1}{\sqrt{(2\pi)^3}}\sqrt{\frac{\hbar c}{2k}}\mathrm{e}^{i\left(\vec{k}\cdot\vec{r}-\omega_k t\right)}\epsilon^y_{\vec{k},\lambda}.
\end{aligned} \tag{26}$$

Combining Eqs. (22)-(26) yields

$$\left[\hat{E}^x, \hat{a}^\dagger_{\vec{k},\lambda}\right]\hat{A}^y|0\rangle = \frac{i\hbar c}{2}\int \frac{d^3k'}{(2\pi)^3}\sqrt{\frac{k}{k'}}\sum_{\lambda'=1,2}\epsilon^x_{\vec{k},\lambda}\epsilon^y_{\vec{k}',\lambda'}\mathrm{e}^{i\left(\left(\vec{k}-\vec{k}'\right)\cdot\vec{r}-(\omega_k-\omega_{k'})t\right)}\hat{a}^\dagger_{\vec{k}',\lambda'}|0\rangle \tag{27}$$

$$\hat{E}^x\left[\hat{A}^y, \hat{a}^\dagger_{\vec{k},\lambda}\right]|0\rangle = -\frac{i\hbar c}{2}\int \frac{d^3k'}{(2\pi)^3}\sqrt{\frac{k}{k'}}\sum_{\lambda'=1,2}\epsilon^y_{\vec{k},\lambda}\epsilon^x_{\vec{k}',\lambda'}\mathrm{e}^{i\left(\left(\vec{k}-\vec{k}'\right)\cdot\vec{r}-(\omega_k-\omega_{k'})t\right)}\hat{a}^\dagger_{\vec{k}',\lambda'}|0\rangle. \tag{28}$$

$$\left[\hat{E}^x\hat{A}^y - \hat{E}^y\hat{A}^x, \hat{a}^\dagger_{\vec{k},\lambda}\right]|0\rangle = i\hbar c\int \frac{d^3k'}{(2\pi)^3}\sqrt{\frac{k}{k'}}\sum_{\lambda'=1,2}\left(\epsilon^x_{\vec{k},\lambda}\epsilon^y_{\vec{k}',\lambda'} - \epsilon^y_{\vec{k},\lambda}\epsilon^x_{\vec{k}',\lambda'}\right)\mathrm{e}^{i\left(\left(\vec{k}-\vec{k}'\right)\cdot\vec{r}-(\omega_k-\omega_{k'})t\right)}\hat{a}^\dagger_{\vec{k}',\lambda'}|0\rangle. \tag{29}$$

Now, according to Eq. (21) we need to perform a volume integration. That would result a Dirac delta via $\int d^3r\mathrm{e}^{i\left(\vec{k}-\vec{k}'\right)\cdot\vec{r}} = (2\pi)^3\,\delta^3\left(\vec{k}-\vec{k}'\right)$, and then, after performing the k' integral, we end up with

$$\begin{aligned}
\hat{S}^z|\vec{k},\lambda\rangle &= i\hbar\sum_{\lambda'=1,2}\left(\epsilon^x_{\vec{k},\lambda}\epsilon^y_{\vec{k},\lambda'} - \epsilon^y_{\vec{k},\lambda}\epsilon^x_{\vec{k},\lambda'}\right)\hat{a}^\dagger_{\vec{k},\lambda'}|0\rangle = i\hbar\sum_{\lambda'=1,2}\left(\epsilon^x_{\vec{k},\lambda}\epsilon^y_{\vec{k},\lambda'} - \epsilon^y_{\vec{k},\lambda}\epsilon^x_{\vec{k},\lambda'}\right)|\vec{k},\lambda'\rangle \\
&= i\hbar\left(\epsilon^x_{\vec{k},\lambda}\epsilon^y_{\vec{k},1} - \epsilon^y_{\vec{k},\lambda}\epsilon^x_{\vec{k},1}\right)|\vec{k},1\rangle + i\hbar\left(\epsilon^x_{\vec{k},\lambda}\epsilon^y_{\vec{k},2} - \epsilon^y_{\vec{k},\lambda}\epsilon^x_{\vec{k},2}\right)|\vec{k},2\rangle.
\end{aligned} \tag{30}$$

Let us write Eq. (30) explicitly for $\lambda = 1, 2$.

$$\hat{S}^z|\vec{k},1\rangle = i\hbar\left(\epsilon^x_{\vec{k},1}\epsilon^y_{\vec{k},2} - \epsilon^y_{\vec{k},1}\epsilon^x_{\vec{k},2}\right)|\vec{k},2\rangle = i\hbar\det\begin{bmatrix}\epsilon^x_{\vec{k},1} & \epsilon^x_{\vec{k},2}\\ \epsilon^y_{\vec{k},1} & \epsilon^y_{\vec{k},2}\end{bmatrix}|\vec{k},2\rangle \overset{(*)}{=} i\hbar|\vec{k},2\rangle$$

$$\hat{S}^z|\vec{k},2\rangle = i\hbar\left(\epsilon^x_{\vec{k},2}\epsilon^y_{\vec{k},1} - \epsilon^y_{\vec{k},2}\epsilon^x_{\vec{k},1}\right)|\vec{k},1\rangle = -i\hbar\det\begin{bmatrix}\epsilon^x_{\vec{k},1} & \epsilon^x_{\vec{k},2}\\ \epsilon^y_{\vec{k},1} & \epsilon^y_{\vec{k},2}\end{bmatrix}|\vec{k},2\rangle \overset{(*)}{=} -i\hbar|\vec{k},1\rangle,$$

where equality (*) is due to the fact that a determinant of an orthogonal matrix is 1.

Interestingly, we see that the *linear* polarized photon is not an eigen-state of $\hat{S}^z$. However, consider the *circular* polarized photon states,

$$|\vec{k},\pm\rangle = \frac{|\vec{k},1\rangle \pm i|\vec{k},2\rangle}{\sqrt{2}}. \tag{31}$$

With this definition, we find that

$$\hat{S}^z|\vec{k},+\rangle = \frac{\hat{S}^z|\vec{k},1\rangle + i\hat{S}^z|\vec{k},2\rangle}{\sqrt{2}} = \frac{i\hbar|\vec{k},2\rangle + \hbar|\vec{k},1\rangle}{\sqrt{2}} = +\hbar|\vec{k},+\rangle \tag{32}$$

$$\hat{S}^z|\vec{k},-\rangle = \frac{\hat{S}^z|\vec{k},1\rangle - i\hat{S}^z|\vec{k},2\rangle}{\sqrt{2}} = \frac{i\hbar|\vec{k},2\rangle - \hbar|\vec{k},1\rangle}{\sqrt{2}} = -\hbar|\vec{k},-\rangle, \tag{33}$$

or

$$\boxed{\hat{S}^z|\vec{k},\pm\rangle = \pm\hbar|\vec{k},\pm\rangle}. \tag{34}$$

Thus, we see that the circular polarized photon is an eigen-state of $\hat{S}^z$ with eigen-values of $m_z = \pm 1$. As these are the only possible states, this proves that the photon is a spin-1 particle!

Note: You may wonder what happened to the $m_z = 0$ state. Since photons are massless they have only two polarizations and therefore only two m_z states. However, you can start with the *Proca* Lagrangian,

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m_A^2 A^\mu A_\mu. \tag{35}$$

This Lagrangian is *not* gauge invariant due to the mass term, and consequently this leads to a third longitudinal polarization that corresponds to $m_z = 0$.