Quantum Mechanics 3 - Homework 2

Due date: 1.12.2022

1. A plane wave with momentum k scatters a potential $V(x)$ in 1D. It is assumed that $V(x) \to 0$ as $x \to \pm\infty$.

   (a) Follow lecture notes 2, and derive the retarded Green function in 1D (it is not the same as the one in 3D!).

   (b) Then write the Lippman-Schwinger equation in 1D, and identify the transmission and reflection coefficients.

   (c) **Bonus:** Show that the transmission and reflection coefficients satisfy $T_k + R_k = 1$. You may use the following property, which is true for every operator $\hat{A}$ and state $|\phi\rangle$,

   $$\langle\phi| \left(\hat{A} - i\epsilon\right)^{-1} |\phi\rangle = \mathcal{P}\langle\phi| \left(\hat{A}\right)^{-1} |\phi\rangle + i\pi\langle\phi|\delta\left(\hat{A}\right)|\phi\rangle. \quad (1)$$

   Here, $\mathcal{P}$ stands for the Cauchy principle value and $\delta(\cdot)$ is the Dirac delta function.

2. A neutral atom with energy E scatters another neutral atom. The Lennard-Jones potential between them reads

   $$V(r) = \epsilon\left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6}\right] \quad (2)$$

   for $r > R$, where R is the minimal distance that the atoms can be in. Consider R to be parametrically smaller than all the length scales in the problem. In the Born approximation, what is the total cross section as a function of the incident energy E?

   Hint: Split the r integral into two regions and make the appropriate approximations to evaluate the integral analytically in the lowest order of qR.

3. (a) Find the phase shift δ_ℓ of the ℓ'th partial wave from an attractive delta-well: $V(r) = -\lambda\delta(r-R)$. Do it *without* using the Born approximation, and do not take the large argument limit for the Hankel function[1]. Plot the result (a program can be used) as a function of k for different values of λ and ℓ.

   (b) Find an equation for the bound states of the delta-well (you don't have to solve that equation!). Do so by solving the radial part of Schrödinger equation.

   Hint: Recall the lovely modified spherical Bessel functions $i_\ell(x)$ and $k_\ell(x)$. In your solution, you may find the following identity useful.

$$i_\ell(x)\,k'_\ell(x) - k_\ell(x)\,i'_\ell(x) = -\frac{\pi}{2x^2}. \tag{3}$$

   (c) Is there a relation between the poles of the scattering amplitude $f_k(\theta,\phi)$ and the bound states you found in the previous item? You may find the following identities useful.

$$i_\ell(x) = i^{-\ell} j_\ell(ix), \qquad k_\ell(x) = -i^\ell h_\ell^{(1)}(ix) \tag{4}$$

[1] It is recommended to review the equations in the beginning of tutorial 4. Trust me, you can find δ_ℓ analytically in this problem.