Quantum Mechanics 3 - Homework 3

Due date: 22.12.2022

1. A particle with mass m is constrained to move along a circle of radius R.

   (a) By using the results you saw in class for a free particle, find the path integral $K_n(\theta_f, t_f; \theta_i, t_i)$ associated with the classical path that winds n times around the circle. Then, express the full path integral along all the possible paths.

   (b) Find the Fourier series in $(\theta_f - \theta_i)$ of the expression you found in (a), namely find the coefficients a_k in

   $$K(\theta_f, t_f; \theta_i, t_i) = \sum_{k=-\infty}^{\infty} a_k \mathrm{e}^{i(\theta_f - \theta_i)k}. \tag{1}$$

   (c) From (b), deduce the energy eigenvalues and eigenfunctions of the system.

2. Using the definition of the propagator, prove the *group property* of propagators,

   $$K(\vec{r}_f, t_f; \vec{r}_i, t_i) = \int d^3r' K(\vec{r}_f, t_f; \vec{r}\,', t') K(\vec{r}\,', t'; \vec{r}_i, t_i), \tag{2}$$

   which is true for any $t_i < t' < t_f$.

3. Time-independent and constant magnetic field acts in the $\hat{z}$ direction, $\vec{B} = B\hat{z}$. Find an expression for the path integral of a particle that starts at $\vec{r}(t_i) = \vec{r}_i$, and end at $\vec{r}(t_f) = \vec{r}_f$.

   Guidance:

   - Recall the Lagrangian for a particle in magnetic field, $L = \frac{1}{2}m\dot{\vec{r}}^2 + q\vec{A}\cdot\vec{v}$. What is the vector potential $\vec{A}$ that corresponds to the magnetic field in this problem? [1]
   - By using the classical EoM, prove that the action in this problem can be decomposed to

   $$S[\vec{r}(t)] = S[\vec{r}_{\text{cl}}(t)] + S[\vec{r}_{\text{qu}}(t)] = S_{\text{cl}} + S_{\text{qu}}, \tag{3}$$

   where $\vec{r}_{\text{cl}}(t)$ is the classical path and $\vec{r}_{\text{qu}}(t)$ are the quantum fluctuations.
   - Calculate S_{cl}, the action of the classical path. Since the calculations are cumbersome, you are encouraged (but not obligated) to use Mathematica.
   - Next, you would need to calculate the QF (quantum fluctuations) part of the propagator. To do so, use the group property (see Problem 2). Use Eqs. (2) and (3) to find the following relation for QF (i.e. find the function $F(t, t', t_i)$)

   $$\text{QF}(t_f, t_i) = \text{QF}(t_f, t')\,\text{QF}(t', t_i) \times F(t, t', t_i). \tag{4}$$

   Now, deduce $\text{QF}(t_f, t_i)$ from Eq. (4). You may find the following trigonometric identity useful,

   $$\tan(\alpha) + \tan(\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha)\cos(\beta)}. \tag{5}$$

4. In this exercise you will study numerically *the stationary phase approximation*. According to this approximation, for any real function $f(t)$, if we define

   $$F(\lambda) := \int_{-\infty}^{\infty} e^{i\lambda f(t)} dt, \tag{6}$$

   then

   $$\lim_{\lambda\to\infty} F(\lambda) = F_{\text{approx}}(\lambda) = \sum_{t_i} \sqrt{\frac{2\pi i}{\lambda f''(t_i)}} e^{i\lambda f(t_i)}, \tag{7}$$

[1] Note that due to gauge invariance, there are actually infinite possible choices for $\vec{A}$. By changing the gauge, $\vec{A} \to \vec{A} + \vec{\nabla}\Lambda$, how does it affect the action?

where $f(t)$ is the second derivative of $f(t)$, and t_i correspond to the extremum of $f(t)$, i.e. $\forall\ t_i,\ f'(t_i) = 0$.

For the following function

$$f(t) = -t^2 + \frac{1}{2}t^4, \tag{8}$$

(a) Calculate the RHS of Eq. (7), $F_{\text{approx}}(\lambda)$.

(b) Use Mathematica[2] to calculate NUMERICALLY $F(\lambda)$ as given by Eq. (6) and $F_{\text{approx}}(\lambda)$ as given by Eq. (7). Plot on the same figure the real and imaginary parts of $F(\lambda)$, for $0 < \lambda \leq 100$. Do the same for $F_{\text{approx}}(\lambda)$.

(c) Now plot $|F(\lambda) - F_{\text{approx}}(\lambda)|$. Write your conclusions.

5. **After tutorial 6:** Using Eq. (3.39) from lecture notes, calculate

$$\langle x_f, t_f | x(t_2)\, x(t_1) | x_i = 0, t_i = 0 \rangle, \qquad t_f > t_2 > t_1 > t_i = 0 \tag{9}$$

for the forced harmonic oscillator with a time dependent force $J(t)$. Your final answer should be expressed with the Green function and integrals over $J(t)$.

6. **After tutorial 6:** Consider a particle of mass m in an infinite well of width a at 1D.

(a) Show that the corresponding propagator is

$$K(x,t;y,0) = \sqrt{\frac{m}{2\pi i\hbar t}} \sum_{n=-\infty}^{\infty} \left\{ \exp\left[\frac{im\,(x-y+2na)^2}{2\hbar t}\right] - \exp\left[\frac{im\,(x+y+2na)^2}{2\hbar t}\right] \right\}. \tag{10}$$

Guidance:

- Start with the semi-classical approximation formula that includes also reflection by hard walls, and use the classical action of a free particle to explain the square root in Eq. (10).

[2]Other programs can be used, but Mathematica is strongly recommended for this exercise.

- There are infinite possible *classical* paths the particle can have to get from y to x within time t, as it can bounce off the walls an arbitrary number of times. Between each successive bounces, the particle can be considered as a free particle. Notice that all the paths can be classified into 4 groups, where each group's paths can be constructed from a fundamental path. Find the 4 the distances associated with the fundamental paths to construct all the other classical paths, and use that result to explain the infinite sum at Eq. (10). Do not forget the Maslov index!

(b) By using the above formula for the propagator, find the energy spectrum. **Hint**: Use the Poisson summation formula:

$$\sum_{n=-\infty}^{\infty} f(\alpha n) = \frac{\sqrt{2\pi}}{\alpha} \sum_{n=-\infty}^{\infty} F\left(\frac{2\pi n}{\alpha}\right), \tag{11}$$

where

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{ikx} dx. \tag{12}$$