Quantum Mechanics 3 - Homework 4

Due date: 12.1.2023

1. Consider the Hamiltonian of a bosonic system,

$$\hat{H} = \hbar\omega\hat{b}^\dagger\hat{b} + \gamma\left(\hat{b}^\dagger + \hat{b}\right), \tag{1}$$

where b, $b^\dagger$ are the bosonic annihilation and creation operators, and ω, γ are constants.

   (a) Find the eigenstates and eigen-energies of the system. Relate the eigenstates basis you found with the Fock basis generated by $b^\dagger$. **Hint:** Define a new operator $\hat{c} = \hat{b} + C$ with some constant C, and find C that diagonalizes the Hamiltonian.

   (b) Do the same for the interacting many-body system of electrons and bosons, where the electrons are created by $\hat{a}_i^\dagger$ and the bosons are created by $\hat{b}_i^\dagger$:

$$\hat{H} = \sum_i \hbar\omega_i\hat{b}_i^\dagger\hat{b}_i + \sum_j \epsilon_j\hat{a}_j^\dagger\hat{a}_j + \sum_{i,j}\gamma_i\hat{a}_j^\dagger\hat{a}_j\left(\hat{b}_i^\dagger + \hat{b}_i\right). \tag{2}$$

   Hint: Does the above trick of shifting the bosonic operator by a constant work here? If not, find the appropriate substitution for which the Hamiltonian can be diagonalized. Note that the bosonic and fermionic operators commute with each other.

2. In class exercise 7 we found that for the Hamiltonian $\hat{H} = \sum_{i,j}^N \epsilon_{ij}\hat{a}_i^\dagger\hat{a}_j$ the partition function is given by

$$Z = \left[\det\left(\mathbb{I} \pm \mathrm{e}^{-\beta\hat{\epsilon}}\right)\right]^{\pm 1}, \tag{3}$$

with $+$ for fermions and $-$ for bosons. Consider the matrix

$$\hat{\epsilon} = \begin{bmatrix} E & -\Delta & 0 & \cdots & \cdots & \cdots & 0 \\ -\Delta & 2E & -\Delta & 0 & \cdots & \cdots & 0 \\ 0 & -\Delta & 3E & -\Delta & 0 & \cdots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & \end{bmatrix}, \tag{4}$$

with $E > 0$, $\Delta = 0.1E$. Write a program that calculates Z for a given β and N (the size of the matrix). Plot $U(\beta)$, where $U = -\partial \ln Z / \partial \beta$ is the thermodynamic energy of the system. The β-axis should be scaled logarithmically and span from $0.01E^{-1}$ to $100E^{-1}$. Show your results for $N = 1, 3, 5$ on the same figure. Do it for both fermions and bosons (two figures in total). Explain your results.

3. **After tutorial 8:** Using Bogoliubov transformation, diagonalize the Hamiltonian

$$\hat{H} = \sum_{k,\sigma} \xi_k \hat{c}^{\dagger}_{k,\sigma} \hat{c}_{k,\sigma} - \sum_{k} \left(\Delta_k \hat{c}^{\dagger}_{k,\uparrow} \hat{c}^{\dagger}_{-k,\downarrow} + \Delta^*_k \hat{c}_{-k,\downarrow} \hat{c}_{k,\uparrow} \right), \tag{5}$$

where $\hat{c}^{\dagger}_{k,\sigma}$ and $\hat{c}_{k,\sigma}$ are the creation and annihilation operators of fermions with momentum k and spin σ. Δ_k is a complex parameter with units of energy and

$$\xi_k = \frac{\hbar^2 k^2}{2m} - E_F, \qquad E_F = \frac{3}{5} \frac{\hbar^2 k_F^2}{2m}. \tag{6}$$

4. **After tutorial 9:** The following refers to tutorial 9:

   (a) Prove Eq. (7); $\left[\hat{\eta}_k, \hat{H}\right] - \Lambda_k \hat{\eta}_k = 0$.

   (b) Prove Eqs. (8)-(9).

   (c) Prove that the string operator $\hat{K}_i$ is unitary. **Hint:** since we know that $\hat{K}_i$ is hermitian, it is sufficient to show $\hat{K}_i^2 = 1$.

   (d) Show that $\hat{c}_i$ and $\hat{c}_i^{\dagger}$ are creation and annihilation operators, namely they satisfy Eqs. (39)-(40).