Quantum Mechanics 3 - Homework 5

Due date: 11.1.2022

1. Lorentz transformations are defined as spacetime transformations that preserve the norm of the vectors,

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu \implies x^\mu x_\mu \to x'^\mu x'_\mu = x^\mu x_\mu. \tag{1}$$

   (a) Recall that a boost transformation along the z-axis is given by the matrix

$$\Lambda^\mu{}_\nu = \begin{bmatrix} \cosh\phi & 0 & 0 & -\sinh\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\phi & 0 & 0 & \cosh\phi \end{bmatrix}, \tag{2}$$

   where ϕ is the rapidity parameter of the boost[1]. For an infinitesimal boost transformation, $\Lambda^\mu{}_\nu = \delta^\mu_\nu + \epsilon^\mu{}_\nu$, show that $\epsilon^{\mu\rho} \equiv \epsilon^\mu{}_\nu \eta^{\nu\rho}$ is an anti-symmetric matrix.

   (b) Repeat the previous item, but now with an infinitesimal rotation transformation around the z-axis.

   (c) Prove that for any $\Lambda^\mu{}_\nu$ that satisfies Eq. (1), the corresponding $\epsilon^{\mu\rho}$ is an anti-symmetric matrix.

2. For the source free EM theory ($J^\mu = 0$), use the Noether theorem to

   (a) Express the Hamiltonian in terms of the $\vec{E}$ and $\vec{B}$ fields.

   (b) Express the momentum of the EM field $\vec{p}$ in terms of the $\vec{E}$ and $\vec{B}$ fields.

   (c) Find an expression for the angular momentum of the EM field $\vec{J}$. Identify the $\vec{L}$ and $\vec{S}$ components. Are they gauge invariant?

[1]In a more familiar notation, $\sinh\phi = \beta$, $\cosh\phi = \gamma\beta$.

(d) By using the result of the previous item, show that

$$\vec{J} = \int d^3x \left[\vec{x} \times \left(\vec{E} \times \vec{B}\right)\right]. \tag{3}$$

3. **After tutorial 11:** For a real massive scalar field, find the momentum $\vec{P}$ with the Noether theorem, and then quantize the field in order to express $\vec{P}$ in terms of annihilation and creation operators.

4. **After tutorial 12:** Compute the momentum of the quantized electromagnetic field in terms of creation and annihilation operators.

   Guidance: In question 2 you found the classical expression for the momentum of the EM field:

$$\vec{P} = \frac{1}{c} \int d^3x \left(\vec{E} \times \vec{B}\right). \tag{4}$$

   However, this is not Hermitian, so consider instead

$$\vec{P} = \frac{1}{2c} \int d^3x \left(\vec{E} \times \vec{B} - \vec{B} \times \vec{E}\right). \tag{5}$$