Quantum Mechanics 3 - Homework 2 Solution

Solution to question 1

The retarded Green function in 3D is defined as ($0 < \epsilon \ll E_k$)

$$G_0\left(\vec{r},\vec{r}'\right) = \langle \vec{r}| \left(E_k - \hat{H}_0 + i\epsilon\right)^{-1} |\vec{r}'\rangle, \tag{1}$$

where $E_k = \hbar^2 k^2/2m$, and $\hat{H}_0 = \hbar^2 \hat{k}^2/2m$, where $\hat{\vec{k}} = \hat{\vec{p}}/\hbar$ and $\hat{\vec{p}}$ is the momentum operator. In 1D it is

$$G_0\left(x,x'\right) = \langle x| \left(E_k - \hat{H}_0 + i\epsilon\right)^{-1} |x'\rangle. \tag{2}$$

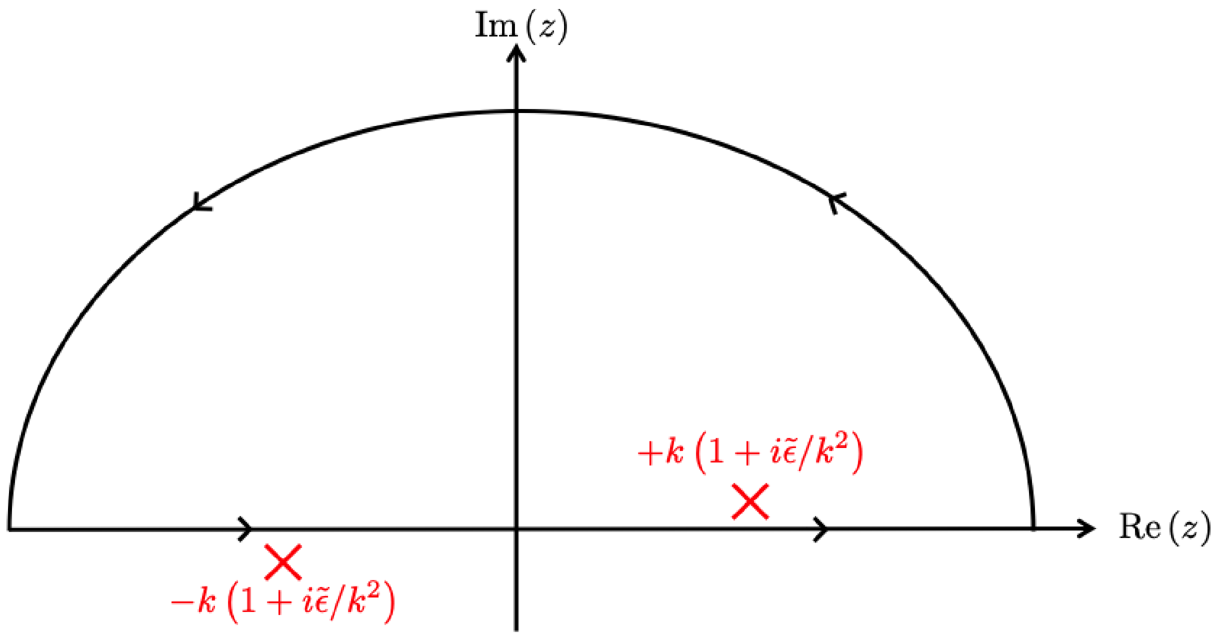
Let us insert the unity operator twice,

$$\begin{aligned}
G_0\left(x,x'\right) &= \langle x \int_{-\infty}^{\infty} dk' |k'\rangle\langle k'| \left(E_k - \hat{H}_0 + i\epsilon\right)^{-1} \int_{-\infty}^{\infty} dk'' |k''\rangle\langle k''|x'\rangle \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk' dk'' \langle x|k'\rangle\langle k''|x'\rangle\langle k'| \left(E_k - \hat{H}_0 + i\epsilon\right)^{-1} |k''\rangle \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk' dk'' \frac{1}{L^{1/2}} \mathrm{e}^{ik'x} \frac{1}{L^{1/2}} \mathrm{e}^{-ik''x'} \underbrace{\langle k'|k''\rangle}_{\frac{L}{2\pi}\delta(k'-k'')} \left(E_k - \hbar^2 k''^2/2m + i\epsilon\right)^{-1} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk' \mathrm{e}^{ik'(x-x')} \left(\hbar^2 k^2/2m - \hbar^2 k'^2/2m + i\epsilon\right)^{-1} \\
&= \frac{m}{\pi\hbar^2}\int_{-\infty}^{\infty} dk' \frac{\mathrm{e}^{ik'(x-x')}}{k^2 - k'^2 + 2i\tilde{\epsilon}} = -\frac{m}{\pi\hbar^2}\int_{-\infty}^{\infty} dk' \frac{\mathrm{e}^{ik'(x-x')}}{k'^2 - (k^2 + 2i\tilde{\epsilon})},
\end{aligned} \tag{3}$$

where in the last step we defined $\tilde{\epsilon} \equiv m\epsilon/\hbar^2$. We can write the last integral as

$$\begin{aligned}
G_0\left(x,x'\right) &= -\frac{m}{\pi\hbar^2}\int_{-\infty}^{\infty} dk' \frac{\mathrm{e}^{ik'(x-x')}}{\left(k' - \sqrt{k^2 + 2i\tilde{\epsilon}}\right)\left(k' + \sqrt{k^2 + 2i\tilde{\epsilon}}\right)} \\
&\approx -\frac{m}{\pi\hbar^2}\int_{-\infty}^{\infty} dk' \frac{\mathrm{e}^{ik'(x-x')}}{\left[k' - k\left(1 + i\tilde{\epsilon}/k^2\right)\right]\left[k' + k\left(1 + i\tilde{\epsilon}/k^2\right)\right]}.
\end{aligned} \tag{4}$$

Let us assume that $x > x'$. In that case, in order to compute the integral, we complete a close contour on the *upper* part of the complex plane and switch to a complex variable $k' \to z$.



Since $x > x'$, the infinite semicircle does not contribute to the integral and we have

$$G_0\left(x, x'\right) = -\frac{m}{\pi\hbar^2} \oint dz \frac{\mathrm{e}^{iz(x-x')}}{\left[z - k\left(1 + i\tilde{\epsilon}/k^2\right)\right]\left[z + k\left(1 + i\tilde{\epsilon}/k^2\right)\right]}.$$

The integrand has two poles, at $z = \pm k\left(1 + i\tilde{\epsilon}/k^2\right)$, but only the one with the $+$ sign is contained inside the contour. Thus, according to the residue theorem,

$$\begin{aligned} G_0\left(x, x'\right) &= -\frac{m}{\pi\hbar^2}\, 2\pi i \ \mathrm{Res}\left\{\frac{\mathrm{e}^{iz(x-x')}}{\left[z - k\left(1 + i\tilde{\epsilon}/k^2\right)\right]\left[z + k\left(1 + i\tilde{\epsilon}/k^2\right)\right]}\right\}_{z=+k(1+i\tilde{\epsilon}/k^2)} \\ &= -i\frac{2m}{\hbar^2}\left[\frac{\mathrm{e}^{iz(x-x')}}{z + k\left(1 + i\tilde{\epsilon}/k^2\right)}\right]_{z=+k(1+i\tilde{\epsilon}/k^2)} = -i\frac{m}{\hbar^2}\frac{\mathrm{e}^{ik\left(1+i\tilde{\epsilon}/k^2\right)(x-x')}}{k\left(1 + i\tilde{\epsilon}/k^2\right)} \underset{\tilde{\epsilon} \ll k^2}{=} -i\frac{m}{\hbar^2}\frac{\mathrm{e}^{ik(x-x')}}{k}. \end{aligned} \quad (5)$$

What happens if $x < x'$? In that case we would complete the contour on the *lower* part of the complex plane, and so in that case the contributing pole would be the one with the $-$ sign. Therefore, we conclude

$$\boxed{G_0\left(x, x'\right) = -i\frac{m}{\hbar^2 k}\mathrm{e}^{ik|x-x'|}}. \quad (6)$$

We now write the Lippman-Schwinger equation in 1D.

$$\begin{aligned} \psi_k(x) &= \frac{1}{L^{1/2}}\mathrm{e}^{ikx} + \int_{-\infty}^{\infty} dx' G_0\left(x, x'\right) V\left(x'\right) \psi_k\left(x'\right) \\ &= \frac{1}{L^{1/2}}\mathrm{e}^{ikx} - i\frac{m}{\hbar^2 k}\int_{-\infty}^{\infty} dx' \mathrm{e}^{ik|x-x'|} V\left(x'\right) \psi_k\left(x'\right) \\ &\underset{x \gg x'}{=} \frac{1}{L^{1/2}}\mathrm{e}^{ikx} - i\frac{m}{\hbar^2 k}\mathrm{e}^{+ikx}\int_{-\infty}^{\infty} dx' \mathrm{e}^{-ikx'} V\left(x'\right) \psi_k\left(x'\right) \\ &= \frac{1}{L^{1/2}}\left[1 - i\frac{mL^{1/2}}{\hbar^2 k}\int_{-\infty}^{\infty} dx' \mathrm{e}^{-ikx'} V\left(x'\right) \psi_k\left(x'\right)\right] \mathrm{e}^{+ikx} \end{aligned} \quad (7)$$

From here, we can identify

$$t_k = 1 - i\frac{mL^{1/2}}{\hbar^2 k}\int_{-\infty}^{\infty} dx' \mathrm{e}^{-ikx'} V\left(x'\right)\psi_k\left(x'\right), \tag{8}$$

and thus the transmission coefficient is

$$\boxed{T_k = \left|t_k\right|^2 = \left|1 - i\frac{mL^{1/2}}{\hbar^2 k}\int_{-\infty}^{\infty} dx' \mathrm{e}^{-ikx'} V\left(x'\right)\psi_k\left(x'\right)\right|^2}. \tag{9}$$

In order to find the reflection coefficient, we take the limit $x \ll x'$ in Eq. (7). This results

$$\psi_k\left(x\right) \underset{x\ll x'}{=} \frac{1}{L^{1/2}}\mathrm{e}^{ikx} - i\frac{m}{\hbar^2 k}\mathrm{e}^{-ikx}\int_{-\infty}^{\infty} dx' \mathrm{e}^{+ikx'} V\left(x'\right)\psi_k\left(x'\right), \tag{10}$$

thus we identify

$$r_k = -i\frac{mL^{1/2}}{\hbar^2 k}\int_{-\infty}^{\infty} dx' \mathrm{e}^{+ikx'} V\left(x'\right)\psi_k\left(x'\right), \tag{11}$$

and the reflection coefficient is

$$\boxed{R_k = \left|r_k\right|^2 = \frac{m^2 L}{\hbar^4 k^2}\left|\int_{-\infty}^{\infty} dx' \mathrm{e}^{+ikx'} V\left(x'\right)\psi_k\left(x'\right)\right|^2}. \tag{12}$$

Bonus: Proof that $T_k + R_k = 1$

Let us define

$$f\left(k, k'\right) \equiv -i\frac{mL^{1/2}}{\hbar^2 k}\int_{-\infty}^{\infty} dx \mathrm{e}^{-ik'x} V\left(x\right)\psi_k\left(x\right) = -i\frac{mL}{\hbar^2 k}\langle k'|\hat{V}|\psi_k\rangle. \tag{13}$$

From Eqs. (8) and (11) we have

$$t_k = 1 + f\left(k, k\right), \qquad r_k = f\left(k, -k\right), \tag{14}$$

so $T_k + R_k = 1$ implies that

$$\left|t_k\right|^2 + \left|r_k\right|^2 = 1 + \left|f\left(k,k\right)\right|^2 + \left|f\left(k,-k\right)\right|^2 + 2\mathrm{Re}\left\{f\left(k,k\right)\right\} \stackrel{!}{=} 1 \tag{15}$$

$$\Longrightarrow \left|f\left(k,k\right)\right|^2 + \left|f\left(k,-k\right)\right|^2 = -2\mathrm{Re}\left\{f\left(k,k\right)\right\}. \tag{16}$$

Thus, it is sufficient to prove Eq. (16).

From the Lippmann-Schwinger equation,

$$|\psi_k\rangle = |k\rangle + \left(E_k - \hat{H}_0 + i\epsilon\right)^{-1}\hat{V}|\psi_k\rangle \quad \Longrightarrow \quad |k\rangle = \left[1 - \left(E_k - \hat{H}_0 + i\epsilon\right)^{-1}\hat{V}\right]|\psi_k\rangle, \tag{17}$$

we have

$$\langle k|\hat{V}|\psi_k\rangle = \langle\psi_k|\left[1 - \left(E_k - \hat{H}_0 + i\epsilon\right)^{-1}\hat{V}\right]^{\dagger}\hat{V}|\psi_k\rangle = \langle\psi_k|\left[1 - \hat{V}\left(E_k - \hat{H}_0 - i\epsilon\right)^{-1}\right]\hat{V}|\psi_k\rangle, \tag{18}$$

where we used the Hermiticity of $\hat{V}$, $\hat{H}_0$ and $\left(\hat{A}^{-1}\right)^\dagger = \left(\hat{A}^\dagger\right)^{-1}$ for $\hat{A} = E_k - \hat{H}_0 + i\epsilon$. Therefore

$$\begin{aligned} -2\mathrm{Re}\left\{f\left(k,k\right)\right\} &= -2\mathrm{Re}\left\{-i\frac{mL}{\hbar^2 k}\langle k|\hat{V}|\psi_k\rangle\right\} = -\frac{2mL}{\hbar^2 k}\mathrm{Im}\left\{\langle k|\hat{V}|\psi_k\rangle\right\} \\ &= -\frac{2mL}{\hbar^2 k}\mathrm{Im}\left\{\langle\psi_k|\left[1-\hat{V}\left(E_k-\hat{H}_0-i\epsilon\right)^{-1}\right]\hat{V}|\psi_k\rangle\right\}. \end{aligned} \tag{19}$$

The first term inside the braces, $\langle\psi_k|\hat{V}|\psi_k\rangle$, is real and has no imaginary part. For the second term, we write

$$\langle\phi_k|\left(E_k-\hat{H}_0-i\epsilon\right)^{-1}|\phi_k\rangle = \mathcal{P}\langle\phi_k|\left(E_k-\hat{H}_0\right)^{-1}|\phi_k\rangle + i\pi\langle\phi_k|\delta\left(E_k-\hat{H}_0\right)|\phi_k\rangle, \tag{20}$$

where $|\phi_k\rangle \equiv V|\psi_k\rangle$ and $\mathcal{P}$ stands for the Cauchy principle value. Since both $\mathcal{P}\langle\phi_k|\left(E_k-\hat{H}_0\right)^{-1}|\phi_k\rangle$ and $\langle\phi_k|\delta\left(E_k-\hat{H}_0\right)|\phi_k\rangle$ are real, we deduce that

$$-2\mathrm{Re}\left\{f\left(k,k\right)\right\} = \frac{2\pi mL}{\hbar^2 k}\langle\phi_k|\delta\left(E_k-\hat{H}_0\right)|\phi_k\rangle. \tag{21}$$

As usual, we insert $1=\int_{-\infty}^{\infty}dk'|k'\rangle\langle k'|$ twice, and use the fact that $\langle k''|\delta\left(E_k-\hat{H}_0\right)|k'\rangle = \frac{L}{2\pi}\delta\left(E_k-\frac{\hbar^2k'^2}{2m}\right)\delta\left(k'-k''\right)$ to eliminate the k'' integral. We also recall that $E_k=\hbar^2k^2/2m$, so

$$\begin{aligned} -2\mathrm{Re}\left\{f\left(k,k\right)\right\} &= \frac{2\pi mL}{\hbar^2 k}\frac{L}{2\pi}\int_{-\infty}^{\infty}dk' \underbrace{\langle\phi_k|k'\rangle}_{=\frac{-i\hbar^2 k}{mL}f^*(k,k')} \delta\left(\frac{\hbar^2}{2m}\left(k^2-k'^2\right)\right) \underbrace{\langle k'|\phi_k\rangle}_{=\frac{i\hbar^2 k}{mL}f(k,k')} \\ &= \frac{\hbar^2 k}{m}\int_{-\infty}^{\infty}dk'\left|f\left(k,k'\right)\right|^2\delta\left(\frac{\hbar^2}{2m}\left(k^2-k'^2\right)\right) \\ &\overset{(*)}{=} \frac{\hbar^2 k}{m}\int_{-\infty}^{\infty}dk'\left|f\left(k,k'\right)\right|^2\frac{\delta\left(k'-k\right)+\delta\left(k'+k\right)}{\left|\frac{\hbar^2k'}{m}\right|} \\ &= \frac{k}{|k|}\left[\left|f\left(k,k\right)\right|^2+\left|f\left(k,-k\right)\right|^2\right] \overset{(**)}{=} \left|f\left(k,k\right)\right|^2+\left|f\left(k,-k\right)\right|^2. \end{aligned} \tag{22}$$

In equality (*) we have used the identity

$$\delta\left(f\left(x\right)\right) = \sum_i \frac{\delta\left(x-x_i\right)}{\left|f'\left(x_i\right)\right|}, \tag{23}$$

where x_i are the roots of $f\left(x\right)$ and $f'\left(x\right)$ is the derivative of $f\left(x\right)$, while in equality (**) we have used $k>0$ since in Eq. (7) we assume that the incident wave propagates to the right. This completes our proof of Eq. (16), and therefore $T_k+R_k=1$.

Solution to question 2

The potential has a spherical symmetry so we can use the following formula for the scattering amplitude,

$$\begin{aligned} f\left(\vec{k}\right) &= -\frac{m}{2\pi\hbar^2}\frac{4\pi}{q}\int_R^{\infty}r\sin\left(qr\right)V\left(r\right)dr = -\frac{2m\epsilon}{\hbar^2 q}\int_R^{\infty}r\sin\left(qr\right)\left[\left(\frac{\sigma}{r}\right)^{12}-\left(\frac{\sigma}{r}\right)^{6}\right]dr \\ &= -\frac{2m\epsilon}{\hbar^2q^3}\int_{qR}^{\infty}x\sin\left(x\right)\left[\left(\frac{q\sigma}{x}\right)^{12}-\left(\frac{q\sigma}{x}\right)^{6}\right]dx, \end{aligned} \tag{24}$$

where $q = 2k\sin(\theta/2)$ and we defined $x \equiv qr$. We now split the integral into two regions,

$$f\left(\vec{k}\right) = -\frac{2m\epsilon}{\hbar^2 q^3}\left\{\int_{qR}^{x_0} x\sin(x)\left[\left(\frac{q\sigma}{x}\right)^{12} - \left(\frac{q\sigma}{x}\right)^{6}\right]dx + \int_{x_0}^{\infty} x\sin(x)\left[\left(\frac{q\sigma}{x}\right)^{12} - \left(\frac{q\sigma}{x}\right)^{6}\right]dx\right\}, \quad (25)$$

which is true for any $x_0 \geq qR$. Let us assume that $qR \ll x_0 \ll 1$, such that we can approximate[1] $\sin(x) \approx x - \frac{x^3}{3!}$. For such a tiny x_0, as the integrand decays very quickly, we can neglect the contribution of the second integral[2], and so

$$\begin{aligned}
f\left(\vec{k}\right) &= -\frac{2m\epsilon}{\hbar^2 q^3}\int_{qR}^{x_0} x\left(x - \frac{x^3}{3!}\right)\left[\left(\frac{q\sigma}{x}\right)^{12} - \left(\frac{q\sigma}{x}\right)^{6}\right]dx \quad (26)\\
&= -\frac{2m\epsilon}{\hbar^2 q^3}\int_{qR}^{x_0}\left[\frac{(q\sigma)^{12}}{x^{10}} - \frac{(q\sigma)^{12}}{6x^8} - \frac{(q\sigma)^6}{x^4} + \frac{(q\sigma)^6}{6x^2}\right]dx\\
&= -\frac{2m\epsilon}{\hbar^2 q^3}\left[-\frac{(q\sigma)^{12}}{9x^9} + \frac{(q\sigma)^{12}}{42x^7} + \frac{(q\sigma)^6}{3x^3} - \frac{(q\sigma)^6}{6x}\right]_{qR}^{x_0}\\
&\underset{qR\ll x_0}{\approx} -\frac{2m\epsilon}{\hbar^2 q^3}\left[-\frac{(q\sigma)^{12}}{9(qR)^9} + \frac{(q\sigma)^{12}}{42(qR)^7} + \frac{(q\sigma)^6}{3(qR)^3} - \frac{(q\sigma)^6}{6(qR)}\right]\\
&\underset{qR\ll 1}{\approx} -\frac{2m\epsilon}{\hbar^2 q^3}\left[-\frac{(q\sigma)^{12}}{9(qR)^9} + \frac{(q\sigma)^{12}}{42(qR)^7}\right] = \frac{2m\epsilon\sigma^{12}}{9\hbar^2 R^9} - \frac{m\epsilon\sigma^{12}}{21\hbar^2 R^9}(qR)^2\\
&= \frac{2m\epsilon\sigma^{12}}{9\hbar^2 R^9}\left[1 - \frac{9}{42}(qR)^2\right]. \quad (27)
\end{aligned}$$

The differential cross section is given by

$$\frac{d\sigma}{d\Omega} = \left|f\left(\vec{k}\right)\right|^2 \approx \Bigg(\underbrace{\frac{2m\epsilon\sigma^{12}}{9\hbar^2 R^9}}_{\equiv\lambda}\Bigg)^2\left[1 - \frac{9}{21}(qR)^2\right] = \lambda^2\left[1 - \frac{12}{7}(kR)^2\sin^2\left(\frac{\theta}{2}\right)\right]. \quad (28)$$

The total cross section is therefore

$$\begin{aligned}
\sigma(E) &= \int\frac{d\sigma}{d\Omega}d\Omega = 2\pi\lambda^2\int_0^{\pi} d\theta\sin\theta\left[1 - \frac{12}{7}(kR)^2\sin^2\left(\frac{\theta}{2}\right)\right]\\
&= 2\pi\lambda^2\left[2 - \frac{12}{7}(kR)^2\underbrace{\int_0^{\pi} d\theta\sin\theta\sin^2\left(\frac{\theta}{2}\right)}_{=1}\right] = 4\pi\lambda^2\left[1 - \frac{6}{7}(kR)^2\right]. \quad (29)
\end{aligned}$$

Since $E = \hbar^2 k^2/2m$ we find that

$$\boxed{\sigma(E) = 4\pi\lambda^2\left(1 - \frac{6}{7}\frac{2mR^2}{\hbar^2}E\right)}. \quad (30)$$

The cross section decreases linearly with the energy. Note that this result is valid only if $E \ll \hbar^2/2mR^2$.

[1]We need to take second lowest order of $\sin(x)$, otherwise the resulting cross section would be independent on the energy.

[2]If you don't believe me, you are very welcome to convince yourself with Mathematica! But make you sure you pick an extremely tiny value for qR.

Solution to question 3

Item 1

On the one hand, we have

$$\begin{aligned} e^{i\delta_\ell}\sin\delta_\ell &= -\frac{2mk}{\hbar^2}\int_0^\infty drr^2 V(r)\, j_\ell(kr)\, R_\ell(kr) = \frac{2mk\lambda}{\hbar^2}\int_0^\infty drr^2\delta(r-R)\, j_\ell(kr)\, R_\ell(kr) \\ &= \frac{2mk\lambda}{\hbar^2}R^2 j_\ell(kR)\, R_\ell(kR) \end{aligned} \tag{31}$$

On the other hand, we have the Lippman-Schwinger equation for the radial part,

$$\begin{aligned} R_\ell(r) &= j_\ell(kr) - \frac{2imk}{\hbar^2}h_\ell^{(1)}(kr)\int_0^\infty dr' r'^2 V(r')\, j_\ell(kr')\, R_\ell(kr') \\ &= j_\ell(kr) + \frac{2imk\lambda}{\hbar^2}h_\ell^{(1)}(kr)\int_0^\infty dr' r'^2\delta(r'-R)\, j_\ell(kr')\, R_\ell(kr') \\ &= j_\ell(kr) + \frac{2imk\lambda}{\hbar^2}h_\ell^{(1)}(kr)\, R^2 j_\ell(kR)\, R_\ell(R), \end{aligned} \tag{32}$$

thus

$$R_\ell(r) = \frac{j_\ell(kr)}{1-\frac{2imk\lambda R^2}{\hbar^2}h_\ell^{(1)}(kr)\, j_\ell(kR)}. \tag{33}$$

By plugging Eq. (33) in (31) we find

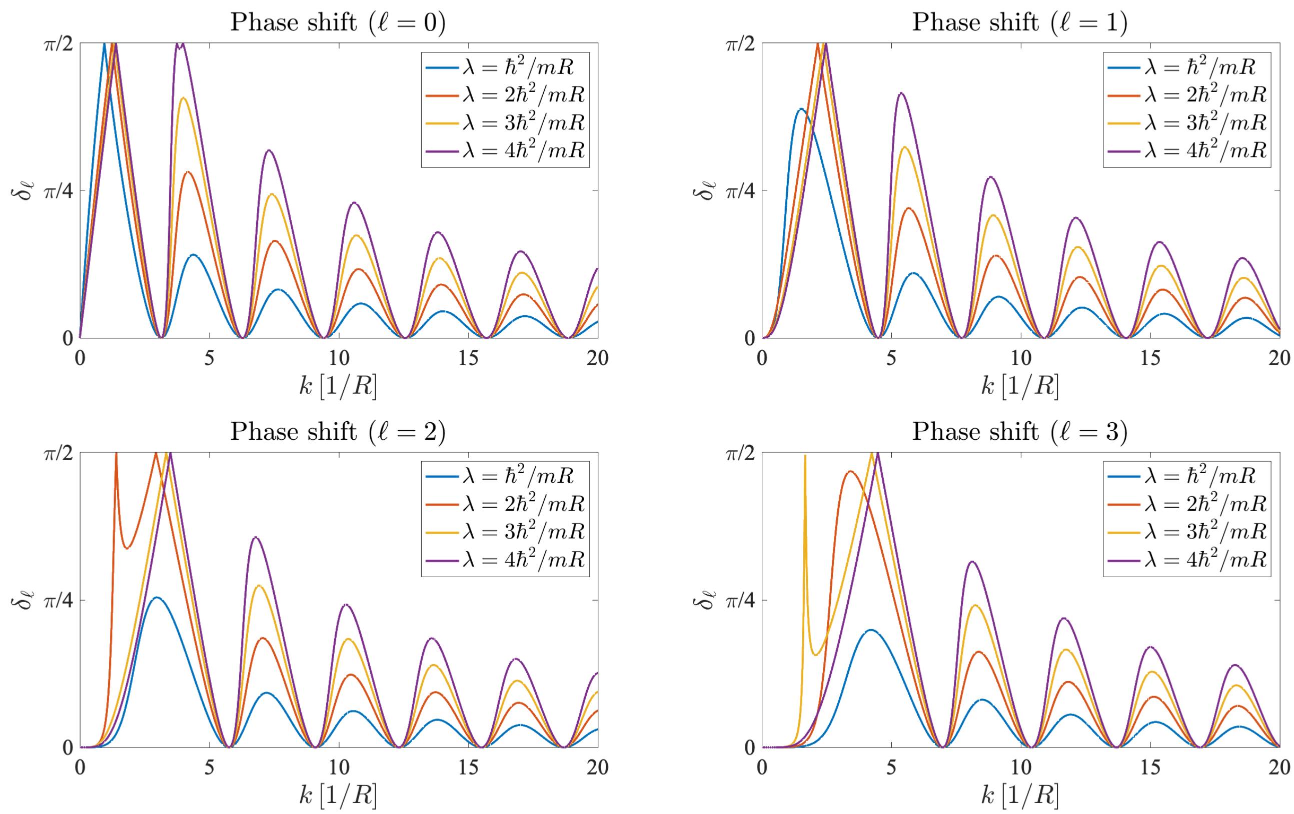
$$e^{i\delta_\ell}\sin\delta_\ell = \frac{\frac{2mk\lambda R^2}{\hbar^2}j_\ell^2(kR)}{1-\frac{2imk\lambda R^2}{\hbar^2}h_\ell^{(1)}(kR)\, j_\ell(kR)}, \tag{34}$$

Now, since $\mathrm{Im}\{e^{i\delta_\ell}\sin\delta_\ell\} = \sin^2\delta_\ell$, we deduce that

$$\delta_\ell = \sin^{-1}\sqrt{\mathrm{Im}\left[\frac{\frac{2mk\lambda R^2}{\hbar^2}j_\ell^2(kR)}{1-\frac{2imk\lambda R^2}{\hbar^2}h_\ell^{(1)}(kR)\, j_\ell(kR)}\right]}. \tag{35}$$

Below we plot[3] the phase shift for several values of ℓ and λ.

[3] Note that there is an ambiguity in determining the exact value of δ_ℓ, as we can only tell the exact value of $\sin^2\delta_\ell$, which is also the observable (as the cross section depends on $\sin^2\delta_\ell$). Therefore, the spikes seen at $\delta_\ell \sim \pi/2$ are only manifestation of this ambiguity.



Item 2

To find bound states, we solve the radial part of the Schrödinger equation (for $E < 0$),

$$\left[\frac{d^2}{dr^2} + \frac{2m}{\hbar^2}\left(E - V\left(r\right)\right) - \frac{\ell\left(\ell+1\right)}{r^2}\right] rR_\ell\left(r\right) = 0, \tag{36}$$

and for $r \neq R$ we get ($K^2 \equiv -2mE/\hbar^2$)

$$\left[\frac{d^2}{dr^2} - K^2 - \frac{\ell\left(\ell+1\right)}{r^2}\right] rR_\ell\left(r\right) = 0, \tag{37}$$

or

$$r^2 R_\ell'' + 2rR_\ell' - \left[K^2 r^2 + \ell\left(\ell+1\right)\right] R_\ell = 0. \tag{38}$$

The solutions to the last equation are the modified spherical Bessel functions, $i_\ell\left(Kr\right)$ and $k_\ell\left(Kr\right)$. Since $\lim\limits_{x\to\infty} i_\ell\left(x\right) = \lim\limits_{x\to 0} k_\ell\left(x\right) = \infty$, we conclude

$$R_\ell\left(r\right) = \begin{cases} A_\ell\, i_\ell\left(Kr\right) & r \leq R \\ B_\ell\, k_\ell\left(Kr\right) & r > R. \end{cases} \tag{39}$$

To find the coefficients A_ℓ and B_ℓ, we demand that the solution is continuous at $r = R$,

$$A_\ell\, i_\ell\left(KR\right) = B_\ell\, k_\ell\left(KR\right) \implies B_\ell = A_\ell \frac{i_\ell\left(KR\right)}{k_\ell\left(KR\right)}. \tag{40}$$

Also, by integrating Eq. (36) from R^- to R^+,

$$R\left[R_\ell'\left(R^+\right) - R_\ell'\left(R^+\right)\right] - \frac{2m}{\hbar^2}\int_{R^-}^{R^+} drV\left(r\right) rR_\ell\left(r\right) = 0, \tag{41}$$

and so by plugging $V\left(r\right) = -\lambda\delta\left(r-R\right)$ we have

$$R_\ell'\left(R^+\right) - R_\ell'\left(R^+\right) = -\frac{2m\lambda}{\hbar^2}R_\ell\left(R\right), \tag{42}$$

and from Eq. (39),

$$A_\ell K\, i_\ell'\left(KR\right) - B_\ell K\, k_\ell'\left(KR\right) = -\frac{2m\lambda}{\hbar^2}A_\ell\, i_\ell\left(KR\right). \tag{43}$$

We now plug B_ℓ from Eq. (40),

$$i_\ell'\left(KR\right) - \frac{i_\ell\left(KR\right)}{k_\ell\left(KR\right)}\, k_\ell'\left(KR\right) = -\frac{2m\lambda}{\hbar^2 K}\, i_\ell\left(KR\right), \tag{44}$$

and after multiplying the equation by $k_\ell\left(KR\right)$,

$$k_\ell\left(KR\right) i_\ell'\left(KR\right) - i_\ell\left(KR\right) k_\ell'\left(KR\right) = -\frac{2m\lambda}{\hbar^2 K}\, i_\ell\left(KR\right) k_\ell\left(KR\right). \tag{45}$$

But since the Wronskian of the modified spherical Bessel functions is

$$i_\ell\left(x\right) k_\ell'\left(x\right) - k_\ell\left(x\right) i_\ell'\left(x\right) = -\frac{\pi}{2x^2}. \tag{46}$$

we find that

$$\frac{\pi}{2K^2R^2} = \frac{2m\lambda}{\hbar^2 K}\, i_\ell\left(KR\right) k_\ell\left(KR\right), \tag{47}$$

or

$$\boxed{1 - \frac{2}{\pi}\frac{2m\lambda KR^2}{\hbar^2}\, i_\ell\left(KR\right) k_\ell\left(KR\right) = 0}. \tag{48}$$

Item 3

Let us now use the identities $i_\ell\left(x\right) = i^{-\ell} j_\ell\left(ix\right)$, and $k_\ell\left(x\right) = -i^\ell h_\ell^{(1)}\left(ix\right)$

$$1 + \frac{2}{\pi}\frac{2m\lambda KR^2}{\hbar^2}\, j_\ell\left(iKR\right) h_\ell^{(1)}\left(iKR\right) = 0. \tag{49}$$

Recall that $k^2 \equiv E^2/2m\hbar^2 = -K^2$. Therefore, we see that $k = iK$ and thus

$$1 - \frac{2}{\pi}\frac{2im\lambda kR^2}{\hbar^2}\, j_\ell\left(kR\right) h_\ell^{(1)}\left(kR\right) = 0. \tag{50}$$

This is (almost!) the denominator of Eq. (34). Recall that $\mathrm{e}^{i\ell}\sin\delta_\ell = f_\ell$, and $S_\ell = 1 + 2if_\ell$. Thus, there is (almost!) a one-to-one correspondence between the poles of the S-matrix and the bound states of the system.