Quantum Mechanics 3 - Homework 3 Solution

Solution to question 1

The propagator of a particle moving on a circle of radius R that winds n times around the circle can be expressed with the following path integral,

$$K_n\left(\theta_f, t_f; \theta_i, t_i\right) = \int_{\theta(t_i)=\theta_i}^{\theta\left(t_f\right)=\theta_f+2\pi n} \mathcal{D}\theta\left(t\right) \mathrm{e}^{\frac{i}{\hbar}\int_{t_i}^{t_f}\frac{1}{2}mR^2\dot{\theta}^2 dt}. \tag{1}$$

We now split the path integral into the classical path and the quantum deviations from it, $\theta = \theta_{\text{cl}} + \theta_{\text{qu}}$,

$$K_n\left(\theta_f, t_f; \theta_i, t_i\right) = \int_{\theta_{\text{qu}}(t_i)=0}^{\theta_{\text{qu}}\left(t_f\right)=0} \mathcal{D}\theta_{\text{qu}}\left(t\right) \mathrm{e}^{\frac{i}{\hbar}\int_{t_i}^{t_f}\frac{1}{2}mR^2\dot{\theta}_{\text{qu}}^2 dt} \times \mathrm{e}^{\frac{i}{\hbar}\int_{t_i}^{t_f}\frac{1}{2}mR^2\dot{\theta}_{\text{cl}}^2 dt} = \sqrt{\frac{mR^2}{2\pi i\hbar\left(t_f - t_i\right)}}\mathrm{e}^{\frac{imR^2\left(\theta_f+2\pi n-\theta_i\right)^2}{2\hbar\left(t_f-t_i\right)}}, \tag{2}$$

where we used the known results for the propagator of a free particle. To account for all the possible windings n, we sum,

$$K\left(\theta_f, t_f; \theta_i, t_i\right) = \sum_{n=-\infty}^{\infty} K_n\left(\theta_f, t_f; \theta_i, t_i\right) = \sqrt{\frac{mR^2}{2\pi i\hbar\left(t_f - t_i\right)}} \sum_{n=-\infty}^{\infty} \mathrm{e}^{\frac{imR^2\left(\theta_f+2\pi n-\theta_i\right)^2}{2\hbar\left(t_f-t_i\right)}}. \tag{3}$$

We now want to find the Fourier series of the propagator,

$$K\left(\theta_f, t_f; \theta_i, t_i\right) = \sum_{k=-\infty}^{\infty} a_k \mathrm{e}^{i\left(\theta_f-\theta_i\right)k} = \sum_{k=-\infty}^{\infty} a_k \mathrm{e}^{i\theta k}. \tag{4}$$

where we defined $\theta \equiv \theta_f - \theta_i$ for convenience. The coefficients a_k are given by the inverse Fourier transform,

$$\begin{aligned}
a_k &= \frac{1}{2\pi}\int_{\theta=0}^{2\pi} d\theta\, K\left(\theta_f, t_f; \theta_i, t_i\right)\mathrm{e}^{-i\theta k} = \frac{1}{2\pi}\int_{\theta=0}^{2\pi} d\theta \sqrt{\frac{mR^2}{2\pi i\hbar\left(t_f-t_i\right)}}\sum_{n=-\infty}^{\infty}\mathrm{e}^{\frac{imR^2(\theta+2\pi n)^2}{2\hbar\left(t_f-t_i\right)}}\mathrm{e}^{-i\theta k} \\
&= \frac{1}{2\pi}\sqrt{\frac{mR^2}{2\pi i\hbar\left(t_f-t_i\right)}}\sum_{n=-\infty}^{\infty}\int_{\theta=0}^{2\pi} d\theta\,\mathrm{e}^{\frac{imR^2(\theta+2\pi n)^2}{2\hbar\left(t_f-t_i\right)}}\mathrm{e}^{-i(\theta+2\pi n)k}\underbrace{\mathrm{e}^{i2\pi nk}}_{=1} \\
&= \frac{1}{2\pi}\sqrt{\frac{mR^2}{2\pi i\hbar\left(t_f-t_i\right)}}\int_{\theta=-\infty}^{\infty} d\theta\,\mathrm{e}^{-\frac{mR^2\theta^2}{2i\hbar\left(t_f-t_i\right)}}\mathrm{e}^{-i\theta k} \\
&= \frac{1}{2\pi}\sqrt{\frac{mR^2}{2\pi i\hbar\left(t_f-t_i\right)}}\sqrt{\frac{2\pi i\hbar\left(t_f-t_i\right)}{mR^2}}\mathrm{e}^{-k^2/\frac{4mR^2}{2i\hbar\left(t_f-t_i\right)}} = \frac{1}{2\pi}\mathrm{e}^{-\frac{i}{\hbar}\frac{\hbar^2k^2}{2mR^2}\left(t_f-t_i\right)},
\end{aligned} \tag{5}$$

where we used the identitiy $\int_{-\infty}^{\infty} d\theta\,\mathrm{e}^{-a\theta^2+b\theta} = \sqrt{\frac{\pi}{a}}\mathrm{e}^{b^2/4a}$, for $a = \frac{mR^2}{2i\hbar\left(t_f-t_i\right)}$ and $b = -ik$.

Thus, we write the propagator as

$$K\left(\theta_f, t_f; \theta_i, t_i\right) = \sum_{k=-\infty}^{\infty} \frac{1}{2\pi} \mathrm{e}^{-\frac{i}{\hbar}\frac{\hbar^2 k^2}{2mR^2}\left(t_f - t_i\right)} \mathrm{e}^{i\left(\theta_f - \theta_i\right)k} \tag{6}$$

Recall that the propagator also satisfies

$$K\left(\theta_f, t_f; \theta_i, t_i\right) = \sum_{k=-\infty}^{\infty} \mathrm{e}^{-\frac{i}{\hbar}E_k\left(t_f - t_i\right)} \psi_k\left(\theta_f\right) \psi_k^*\left(\theta_i\right), \tag{7}$$

where E_k and $\psi_k\left(\theta\right)$ are the eigen-energies and eigen-functions, respectively. By comparing Eq. (6) and (7) we immediately identify

$$\boxed{E_k = \frac{\hbar^2 k^2}{2mR^2}, \qquad \psi_k\left(\theta\right) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{i\theta k}}, \tag{8}$$

as expected.

Solution to question 2

The propagator is defined by

$$K\left(\vec{r}_f, t_f; \vec{r}_i, t_i\right) \equiv \langle \vec{r}_f | \hat{U}\left(t_f, t_i\right) | \vec{r}_i \rangle. \tag{9}$$

We use the composition property of the evolution operator, $\hat{U}\left(t_f, t_i\right) = \hat{U}\left(t_f, t'\right)\hat{U}\left(t', t_i\right)$, which is true for any t' between $t_i < t' < t_f$,

$$K\left(\vec{r}_f, t_f; \vec{r}_i, t_i\right) = \langle \vec{r}_f | \hat{U}\left(t_f, t'\right)\hat{U}\left(t', t_i\right) | \vec{r}_i \rangle. \tag{10}$$

Next, we insert $1 = \int d^3r' |\vec{r}'\rangle\langle\vec{r}'|$,

$$\begin{aligned} K\left(\vec{r}_f, t_f; \vec{r}_i, t_i\right) &= \langle \vec{r}_f | \hat{U}\left(t_f, t'\right) \int d^3r' |\vec{r}'\rangle\langle\vec{r}'| \hat{U}\left(t', t_i\right) | \vec{r}_i \rangle \\ &= \int d^3r' \langle \vec{r}_f | \hat{U}\left(t_f, t'\right) |\vec{r}'\rangle\langle\vec{r}'| \hat{U}\left(t', t_i\right) | \vec{r}_i \rangle \\ &= \int d^3r' K\left(\vec{r}_f, t_f; \vec{r}', t'\right) K\left(\vec{r}', t'; \vec{r}_i, t_i\right). \end{aligned} \tag{11}$$

Solution to question 3

The action

The Lagrangian of a particle in the presence of a magnetic field,

$$L = \frac{1}{2}m\dot{\vec{r}}^2 + q\vec{A}\cdot\vec{v}. \tag{12}$$

We can obtain a magnetic field $\vec{B} = B\hat{z}$ by choosing[1] $\vec{A} = \frac{1}{2}Bx\hat{y} - \frac{1}{2}By\hat{x}$. Thus

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{1}{2}qB\left(x\dot{y} - y\dot{x}\right). \tag{13}$$

[1] Every choice that satisfies $\vec{B} = B\hat{z} = \vec{\nabla}\times\vec{A}$ is acceptable. However, working with a different gauge (such as $\vec{A} = Bx\hat{y}$) would lead to an extra constant term in the action (which has no physical significance!). We chose to work here with the antisymmetric gauge as it yields the simplest action.

The classical equations of motion are

$$\frac{d}{dt}\frac{dL}{d\dot{x}_{\rm cl}}-\frac{dL}{dx_{\rm cl}}=0 \qquad \Longrightarrow \qquad m\ddot{x}_{\rm cl}-qB\dot{y}_{\rm cl}=0 \tag{14}$$

$$\frac{d}{dt}\frac{dL}{d\dot{y}_{\rm cl}}-\frac{dL}{dy_{\rm cl}}=0 \qquad \Longrightarrow \qquad m\ddot{y}_{\rm cl}+qB\dot{x}_{\rm cl}=0 \tag{15}$$

$$\frac{d}{dt}\frac{dL}{d\dot{z}_{\rm cl}}-\frac{dL}{dz_{\rm cl}}=0 \qquad \Longrightarrow \qquad m\ddot{z}_{\rm cl}=0. \tag{16}$$

The action for the Lagrangian in Eq. (13) is

$$\begin{aligned} S &= \int L dt=\int\left[\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)+\frac{1}{2}qB\left(x\dot{y}-y\dot{x}\right)\right]dt \\ &= \int\left[\frac{1}{2}m\left(\left(\dot{x}_{\rm cl}+\dot{x}_{\rm qu}\right)^2+\left(\dot{y}_{\rm cl}+\dot{y}_{\rm qu}\right)^2+\left(\dot{z}_{\rm cl}+\dot{z}_{\rm qu}\right)^2\right)+\frac{1}{2}qB\left(\left(x_{\rm cl}+x_{\rm qu}\right)\left(\dot{y}_{\rm cl}+\dot{y}_{\rm qu}\right)-\left(y_{\rm cl}+y_{\rm qu}\right)\left(\dot{x}_{\rm cl}+\dot{x}_{\rm qu}\right)\right)\right]dt \\ &= \int\Bigg[\frac{1}{2}m\left(\dot{x}_{\rm cl}^2+\dot{y}_{\rm cl}^2+\dot{z}_{\rm cl}^2\right)+\frac{1}{2}qB\left(x_{\rm cl}\dot{y}_{\rm cl}-y_{\rm cl}\dot{x}_{\rm cl}\right) \\ &\quad +\frac{1}{2}m\left(\dot{x}_{\rm qu}^2+\dot{y}_{\rm qu}^2+\dot{z}_{\rm qu}^2\right)+\frac{1}{2}qB\left(x_{\rm qu}\dot{y}_{\rm qu}-y_{\rm qu}\dot{x}_{\rm qu}\right) \\ &\quad +m\left(\dot{x}_{\rm cl}\dot{x}_{\rm qu}+\dot{y}_{\rm cl}\dot{y}_{\rm qu}+\dot{z}_{\rm cl}\dot{z}_{\rm qu}\right)+\frac{1}{2}qB\left(x_{\rm qu}\dot{y}_{\rm cl}+x_{\rm cl}\dot{y}_{\rm qu}-y_{\rm qu}\dot{x}_{\rm cl}-y_{\rm cl}\dot{x}_{\rm qu}\right)\Bigg]dt. \end{aligned} \tag{17}$$

The first line in Eq. (17) is the classical action, $S_{\rm cl}$, while the second line is the contribution of the quantum fluctuations to the action, $S_{\rm qu}$. The third line contains mixed terms. Let us look more closely at this line.

$$\begin{aligned} & \int\left[m\left(\dot{x}_{\rm cl}\dot{x}_{\rm qu}+\dot{y}_{\rm cl}\dot{y}_{\rm qu}+\dot{z}_{\rm cl}\dot{z}_{\rm qu}\right)+\tfrac{1}{2}qB\left(x_{\rm qu}\dot{y}_{\rm cl}+x_{\rm cl}\dot{y}_{\rm qu}-y_{\rm qu}\dot{x}_{\rm cl}-y_{\rm cl}\dot{x}_{\rm qu}\right)\right]dt \\ = & \int\left[m\left(-\ddot{x}_{\rm cl}x_{\rm qu}-\ddot{y}_{\rm cl}y_{\rm qu}-\ddot{z}_{\rm cl}z_{\rm qu}\right)+\tfrac{1}{2}qB\left(x_{\rm qu}\dot{y}_{\rm cl}-\dot{x}_{\rm cl}y_{\rm qu}-y_{\rm qu}\dot{x}_{\rm cl}+\dot{y}_{\rm cl}x_{\rm qu}\right)\right]dt \\ = & -\int\left[\left(m\ddot{x}_{\rm cl}-qB\dot{y}_{\rm cl}\right)x_{\rm qu}+\left(m\ddot{y}_{\rm cl}+qB\dot{x}_{\rm cl}\right)y_{\rm qu}+m\ddot{z}_{\rm cl}z_{\rm qu}\right]dt=0. \end{aligned} \tag{18}$$

Here we used integration by parts and the last equality follows the classical equations of motion (Eqs. 14-16). Therefore, we have

$$\boxed{S=S_{\rm cl}+S_{\rm qu}}. \tag{19}$$

The classical action

In order to find $S_{\rm cl}$ we need to solve the classical equations of motion. The solution to Eq. (16) is the solution for a free particle propagating along the z-axis,

$$z_{\rm cl}\left(t\right)=z_i+\frac{z_f-z_i}{t_f-t_i}t. \tag{20}$$

In order to find $x_{\rm cl}\left(t\right)$ and $y_{\rm cl}\left(t\right)$ we take the derivative of Eq. (14), and plug in it the expression for $\ddot{y}_{\rm cl}$ from Eq. (15). This gives

$$\dddot{x}_{\rm cl}+\frac{q^2B^2}{m^2}\dot{x}_{\rm cl}=0. \tag{21}$$

The solution for this equation is

$$\dot{x}_{\rm cl}\left(t\right)=a\cos\left(\omega t\right)+b\sin\left(\omega t\right), \tag{22}$$

where $\omega \equiv qB/m$ is the cyclotron frequency, and thus

$$x_{\rm cl}(t) = \frac{a}{\omega}\sin(\omega t) - \frac{b}{\omega}\cos(\omega t) + c. \tag{23}$$

From Eq. (14),

$$\dot{y}_{\rm cl}(t) = \frac{1}{\omega}\ddot{x}_{\rm cl}(t) = -a\sin(\omega t) + b\cos(\omega t), \tag{24}$$

and thus

$$y_{\rm cl}(t) = \frac{a}{\omega}\cos(\omega t) + \frac{b}{\omega}\sin(\omega t) + d. \tag{25}$$

We have four unknowns (a, b, c and d) and four boundary conditions,

$$x_i = \frac{a}{\omega}\sin(\omega t_i) - \frac{b}{\omega}\cos(\omega t_i) + c \tag{26}$$

$$x_f = \frac{a}{\omega}\sin(\omega t_f) - \frac{b}{\omega}\cos(\omega t_f) + c \tag{27}$$

$$y_i = \frac{a}{\omega}\cos(\omega t_i) + \frac{b}{\omega}\sin(\omega t_i) + d \tag{28}$$

$$y_f = \frac{a}{\omega}\cos(\omega t_f) + \frac{b}{\omega}\sin(\omega t_f) + d. \tag{29}$$

Or, in matrix form,

$$\begin{bmatrix} \sin(\omega t_i)/\omega & -\cos(\omega t_i)/\omega & 1 & 0 \\ \sin(\omega t_f)/\omega & -\cos(\omega t_f)/\omega & 1 & 0 \\ \cos(\omega t_i)/\omega & \sin(\omega t_i)/\omega & 0 & 1 \\ \cos(\omega t_f)/\omega & \sin(\omega t_f)/\omega & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} x_i \\ x_f \\ y_i \\ y_f \end{bmatrix}. \tag{30}$$

The scary 4×4 matrix can be inverted[2] and one finds that

$$a = \omega \frac{[\sin(\omega t_f) - \sin(\omega t_i)](x_f - x_i) + [\cos(\omega t_f) - \cos(\omega t_i)](y_f - y_i)}{4\sin^2[\omega(t_f - t_i)/2]} \tag{31}$$

$$b = -\omega \frac{[\cos(\omega t_f) - \cos(\omega t_i)](x_f - x_i) - [\sin(\omega t_f) - \sin(\omega t_i)](y_f - y_i)}{4\sin^2[\omega(t_f - t_i)/2]} \tag{32}$$

$$c = \frac{x_f + x_i}{2} + \frac{y_f - y_i}{2}\cot\left[\frac{\omega(t_f - t_i)}{2}\right] \tag{33}$$

$$d = \frac{y_f + y_i}{2} - \frac{x_f - x_i}{2}\cot\left[\frac{\omega(t_f - t_i)}{2}\right]. \tag{34}$$

The rest of the calculation involves really messy algebra, but one can show[3] that the classical action is given by

$$\boxed{S_{\rm cl} = \frac{m\omega}{4}\left\{\cot\left[\frac{\omega(t_f - t_i)}{2}\right]\left[(x_f - x_i)^2 + (y_f - y_i)^2\right] + 2(x_i y_f - x_f y_i)\right\} + \frac{m(z_f - z_i)^2}{2(t_f - t_i)}}. \tag{35}$$

Note: as a quick sanity check we can take the limit where $B \to 0$ (or $\omega \to 0$). In this limit we have a free particle wandering in 3D, so we expect to get the classical action for a free particle in 3D. This is precisely what happens because $\lim_{\omega \to 0} \omega\cot[\omega(t_f - t_i)/2] = \frac{2}{(t_f - t_i)}$.

[2]Mathematica was very helpful here.
[3]Again, with Mathematica.

The propagator

Note that the decomposition of the action into a classical contribution and a quantum contribution (Eq. 19) is not possible in the general case! Because of this neat property, the propagator can be also decomposed to a classical contribution and a quantum contribution,

$$K\left(\vec{r}_f,t_f;\vec{r}_i,t_i\right)=\int_{\vec{r}(t=t_i)=\vec{r}_i}^{\vec{r}\left(t=t_f\right)=\vec{r}_f} D\vec{r}\left(t\right)\mathrm{e}^{iS/\hbar}=\mathrm{e}^{iS_{\mathrm{cl}}/\hbar}\underbrace{\int_{\vec{r}_{\mathrm{qu}}(t=t_i)=0}^{\vec{r}_{\mathrm{qu}}\left(t=t_f\right)=0} D\vec{r}_{\mathrm{qu}}\left(t\right)\mathrm{e}^{iS_{\mathrm{qu}}/\hbar}}_{\mathrm{QF}}, \tag{36}$$

where QF stands for Quantum Fluctuations. In order to find QF, we use the group property of propagators,

$$K\left(\vec{r}_f,t_f;\vec{r}_i,t_i\right)=\int d^3r' K\left(\vec{r}_f,t_f;\vec{r}\,',t'\right)K\left(\vec{r}\,',t';\vec{r}_i,t_i\right). \tag{37}$$

Note that the factor QF does not depend on the locations of the end points in space, only on their time difference. Therefore, the QF factors in Eq. (37) can be placed outside the $\vec{r}\,'$ integral when we plug Eq. (36) in Eq. (37).

$$\mathrm{QF}\left(t_f,t_i\right)\mathrm{e}^{\frac{i}{\hbar}S_{\mathrm{cl}}\left(\vec{r}_f,t_f;\vec{r}_i,t_i\right)}=\mathrm{QF}\left(t_f,t'\right)\mathrm{QF}\left(t',t_i\right)\int d^3r'\mathrm{e}^{\frac{i}{\hbar}S_{\mathrm{cl}}\left(\vec{r}_f,t_f;\vec{r}\,',t'\right)}\mathrm{e}^{\frac{i}{\hbar}S_{\mathrm{cl}}\left(\vec{r}\,',t';\vec{r}_i,t_i\right)}. \tag{38}$$

This is especially true for $\vec{r}_f=\vec{r}_i=\vec{0}$, where $S_{\mathrm{cl}}\left(\vec{0},t_f;\vec{0},t_i\right)=0$,

$$\mathrm{QF}\left(t_f,t_i\right)=\mathrm{QF}\left(t_f,t'\right)\mathrm{QF}\left(t',t_i\right)\int d^3r'\mathrm{e}^{\frac{i}{\hbar}\left[S_{\mathrm{cl}}\left(\vec{0},t_f;\vec{r}\,',t'\right)+S_{\mathrm{cl}}\left(\vec{r}\,',t';\vec{0},t_i\right)\right]}. \tag{39}$$

We now solve the volume integral. We already know that the free propagation along the z-axis gives a factor of $\mathrm{QF}_z=\sqrt{\frac{m}{2\pi i\hbar\left(t_f-t_i\right)}}$ to QF, so we only focus on the contribution that comes from the $x-y$ plane.

$$\begin{aligned}\int d^2r'\mathrm{e}^{\frac{i}{\hbar}\left[S_{\mathrm{cl}}\left(\vec{0},t_f;\vec{r}\,',t'\right)+S_{\mathrm{cl}}\left(\vec{r}\,',t';\vec{0},t_i\right)\right]} &= \int d^2r'\exp\left\{-\frac{m\omega}{4i\hbar}\left(\cot\left[\frac{\omega\left(t_f-t'\right)}{2}\right]+\cot\left[\frac{\omega\left(t'-t_i\right)}{2}\right]\right)\left(x'^2+y'^2\right)\right\}\\ &= \frac{4\pi i\hbar}{m\omega}\left(\cot\left[\frac{\omega\left(t_f-t'\right)}{2}\right]+\cot\left[\frac{\omega\left(t'-t_i\right)}{2}\right]\right)^{-1}\\ &= \frac{4\pi i\hbar}{m\omega}\frac{\tan\left[\frac{\omega\left(t_f-t'\right)}{2}\right]\tan\left[\frac{\omega\left(t'-t_i\right)}{2}\right]}{\tan\left[\frac{\omega\left(t_f-t'\right)}{2}\right]+\tan\left[\frac{\omega\left(t'-t_i\right)}{2}\right]}\\ &= \frac{4\pi i\hbar}{m\omega}\frac{\sin\left[\frac{\omega\left(t_f-t'\right)}{2}\right]\sin\left[\frac{\omega\left(t'-t_i\right)}{2}\right]}{\sin\left[\frac{\omega\left(t_f-t_i\right)}{2}\right]},\end{aligned} \tag{40}$$

where in the last equality we used the trigonometric identity $\tan\left(\alpha\right)+\tan\left(\beta\right)=\sin\left(\alpha+\beta\right)/\cos\left(\alpha\right)\cos\left(\beta\right)$. Therefore, we have (for the contribution of the $x-y$ plane to QF)

$$\frac{m\omega}{4\pi i\hbar}\mathrm{QF}_{xy}\left(t_f,t_i\right)\sin\left[\frac{\omega\left(t_f-t_i\right)}{2}\right]=\mathrm{QF}_{xy}\left(t_f,t'\right)\sin\left[\frac{\omega\left(t_f-t'\right)}{2}\right]\times\mathrm{QF}_{xy}\left(t',t_i\right)\sin\left[\frac{\omega\left(t'-t_i\right)}{2}\right]. \tag{41}$$

From the last equation, it should be clear that the most obvious solution is

$$\mathrm{QF}_{xy}\left(t_f,t_i\right)=\frac{m\omega}{4\pi i\hbar\sin\left[\frac{\omega\left(t_f-t_i\right)}{2}\right]}=\frac{m}{2\pi i\hbar\left(t_f-t_i\right)}\frac{\frac{\omega\left(t_f-t_i\right)}{2}}{\sin\left[\frac{\omega\left(t_f-t_i\right)}{2}\right]}. \tag{42}$$

Combining everything together, we conclude that the propagator is

$$\boxed{K\left(\vec{r}_f, t_f; \vec{r}_i, t_i\right) = \left(\frac{m}{2\pi i\hbar\left(t_f - t_i\right)}\right)^{3/2} \frac{\frac{\omega\left(t_f - t_i\right)}{2}}{\sin\left[\frac{\omega\left(t_f - t_i\right)}{2}\right]} \mathrm{e}^{iS_{\mathrm{cl}}/\hbar}}. \tag{43}$$

Note: since $\lim_{x\to 0} x/\sin(x) = 1$, we see again that taking the limit $\omega \to 0$ results the propagator of a free particle in 3D.

Alternative Way

An alternative way (which I think is more elegant and requires less effort) to find the QF factor would be the following. In the semi-classical approximation, in 3D, the propagator is given by

$$K_{\mathrm{scl}}\left(\vec{r}_f, t_f; \vec{r}_i, t_i\right) = \left[\det\left(\frac{i}{2\pi\hbar} \frac{\partial^2 S_{\mathrm{cl}}}{\partial r_f^i \partial r_i^j}\right)\right]^{1/2} \mathrm{e}^{\frac{i}{\hbar} S_{\mathrm{cl}}\left(\vec{r}_f, t_f; \vec{r}_i, t_i\right)}, \tag{44}$$

where $r_i^j = [x_i, y_i, z_i]$, $r_f^j = [x_f, y_f, z_f]$. The z-axis contribution results the known propagator of a free particle in 1D, so we shall focus only on the $x - y$ plane. We can identify from the above expression that

$$\mathrm{QF}_{xy}^{(\mathrm{scl})}\left(t_f, t_i\right) = \left[\det\left(\frac{i}{2\pi\hbar} \frac{\partial^2 S_{\mathrm{cl}}}{\partial r_f^i \partial r_i^j}\right)\right]^{1/2} \tag{45}$$

where $\mathrm{QF}_{xy}^{(\mathrm{scl})}$ are the quantum fluctuations at the $x{-}y$ plane that contribute to the propagator <u>in the semi-classical approximation</u>. The second derivatives of the classical action (Eq. 35) are

$$\frac{\partial^2 S_{\mathrm{cl}}}{\partial x_f \partial x_i} = \frac{\partial^2 S_{\mathrm{cl}}}{\partial y_f \partial y_i} = -\frac{m\omega}{2} \cot\left[\frac{\omega\left(t_f - t_i\right)}{2}\right], \qquad \frac{\partial^2 S_{\mathrm{cl}}}{\partial y_f \partial x_i} = -\frac{\partial^2 S_{\mathrm{cl}}}{\partial x_f \partial y_i} = \frac{m\omega}{2} \tag{46}$$

and thus

$$\begin{aligned} \mathrm{QF}_{xy}^{(\mathrm{scl})}\left(t_f, t_i\right) &= \left[\left(\frac{i}{2\pi\hbar}\right)^2 \frac{\partial^2 S_{cl}}{\partial x_f \partial x_i} \frac{\partial^2 S_{cl}}{\partial y_f \partial y_i} - \left(\frac{i}{2\pi\hbar}\right)^2 \frac{\partial^2 S_{cl}}{\partial y_f \partial x_i} \frac{\partial^2 S_{cl}}{\partial x_f \partial y_i}\right]^{1/2} \\ &= \left[\left(\frac{-i}{2\pi\hbar} \frac{m\omega}{2}\right)^2 \cot^2\left[\frac{\omega\left(t_f - t_i\right)}{2}\right] + \left(\frac{-i}{2\pi\hbar} \frac{m\omega}{2}\right)^2\right]^{1/2} \\ &= \frac{m\omega}{4\pi i\hbar} \left(\cot^2\left[\frac{\omega\left(t_f - t_i\right)}{2}\right] + 1\right)^{1/2} = \frac{m\omega}{4\pi i\hbar \sin\left[\frac{\omega\left(t_f - t_i\right)}{2}\right]}. \end{aligned} \tag{47}$$

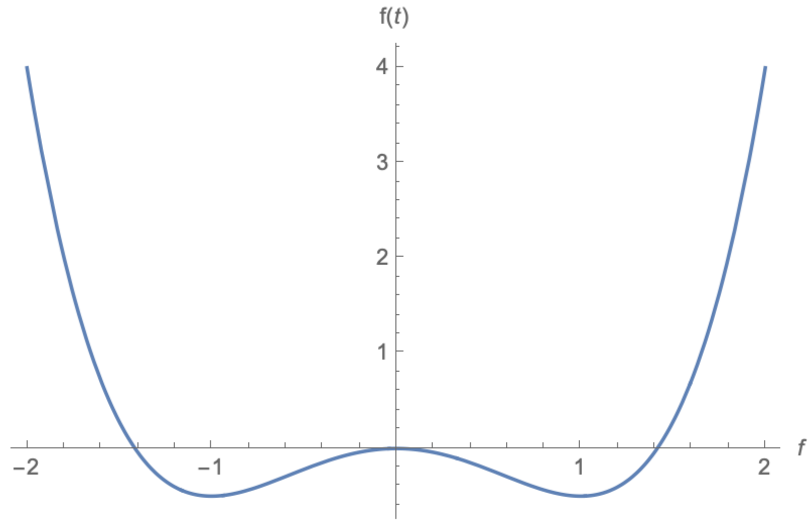
So we found the expression for $\mathrm{QF}_{xy}^{(\mathrm{scl})}$. I now need to convince you that $\mathrm{QF}_{xy} = \mathrm{QF}_{xy}^{(\mathrm{scl})}$, i.e. that the quantum fluctuations derived in the semi-classical approximation are the *exact* quantum fluctuations. But this is true because of Eq. (19); in any system where the action can be written as in Eq. (19), $K = K_{\mathrm{scl}}$, and hence $\mathrm{QF} = \mathrm{QF}^{(\mathrm{scl})}$ (convince yourself why this is true!). So we conclude

$$\mathrm{QF}_{xy}\left(t_f, t_i\right) = \frac{m\omega}{4\pi i\hbar \sin\left[\frac{\omega\left(t_f - t_i\right)}{2}\right]}, \tag{48}$$

in agreement with Eq. (42).

Solution to question 4

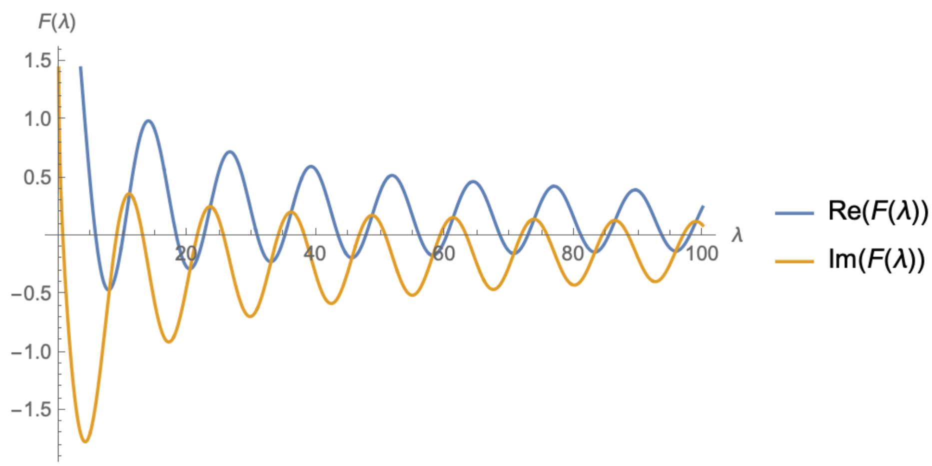
The function $f(t) = -t^2 + \frac{1}{2}t^4$ is presented below.

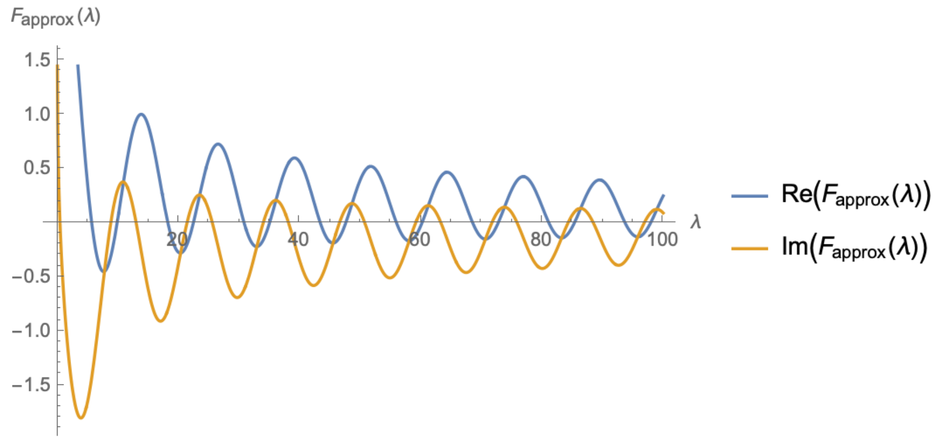


This function has extremum points at $t_i = 0, \pm 1$. By using the stationary phase approximation, we find

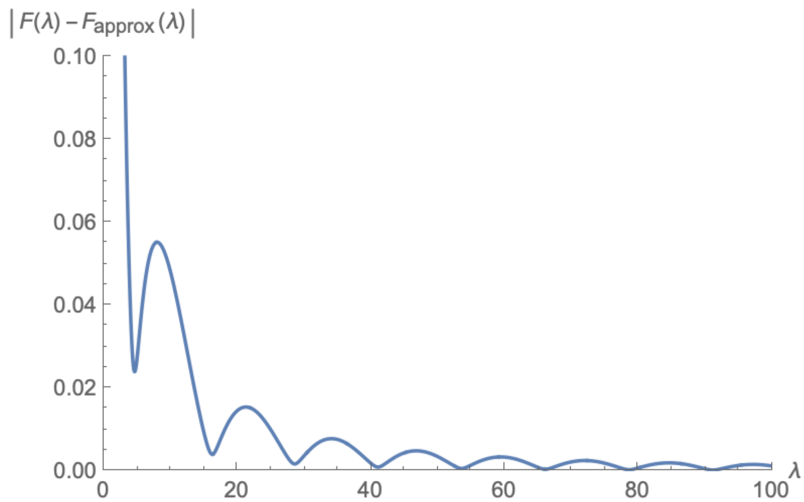
$$F_{\text{approx}}(\lambda) = \sqrt{\frac{2\pi i}{-2\lambda}} + 2\sqrt{\frac{2\pi i}{4\lambda}} \mathrm{e}^{-i\lambda/2} \tag{49}$$

We now plot the real and imaginary parts of $F(\lambda)$ and $F_{\text{approx}}(\lambda)$.





Interestingly, the numerical integral and the approximated expression seem to agree even for small values of λ. Yet, a better examination would include looking at their difference, which is what we do next.



Although the difference oscillates we can observe a decreasing trend as λ increases. For every value of λ, the difference is much smaller than the integral's value, and this is especially true for $\lambda \gtrsim 50$, implying that the stationary phase approximation indeed works.

Solution to question 5

Using the method seen in the tutorial for calculating correlation functions,

$$\begin{aligned}\langle x_f, t_f | x(t_2)\, x(t_1) | x_i = 0, t_i = 0 \rangle &= \int_{x(t=0)=0}^{x(t=t_f)=x_f} \mathcal{D}x(t)\, x(t_1)\, x(t_2)\, \mathrm{e}^{\frac{i}{\hbar} S_{\text{cl}}} \\ &= \int_{x(t=0)=0}^{x(t=t_f)=x_f} \mathcal{D}x(t)\, x(t_1)\, x(t_2) \exp\left\{ \frac{i}{\hbar} \int_0^{t_f} \left[\frac{1}{2} m\dot{x}^2 - \frac{1}{2} m\omega^2 x^2 + J(t)\, x(t) \right] dt \right\},\end{aligned} \tag{50}$$

and as we saw, we can evaluate this expression by taking functional derivatives,

$$\langle x_f, t_f | x(t_2) x(t_1) | x_i = 0, t_i = 0 \rangle = \left(\frac{\hbar}{i}\right)^2 \frac{\delta^2}{\delta J(t_1) \delta J(t_2)} \int_{x(t=0)=0}^{x(t=t_f)=x_f} \mathcal{D}x(t) \exp\left\{\frac{i}{\hbar}\int_0^{t_f}\left[\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2 + J(t)\,x(t)\right] dt\right\} \tag{51}$$

The Lagrangian in this problem is

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2 + Jx, \tag{52}$$

and the classical EoM is given by

$$\frac{d}{dt}\frac{\partial L_{\rm cl}}{\partial \dot{x}} - \frac{\partial L_{\rm cl}}{\partial x} = 0 \qquad \Longrightarrow \qquad m\ddot{x}_{\rm cl} + m\omega^2 x_{\rm cl} - J = 0. \tag{53}$$

We now write the Lagrangian with $x = x_{\rm cl} + y$,

$$\begin{aligned} L \;=\; & \frac{1}{2}m\dot{x}_{\rm cl}^2 - \frac{1}{2}m\omega^2 x_{\rm cl}^2 + Jx_{\rm cl} \\ & + \frac{1}{2}m\dot{y}^2 - \frac{1}{2}m\omega^2 y^2, \\ & + m\dot{x}_{\rm cl}\dot{y} - m\omega^2 x_{\rm cl} y + Jy, \end{aligned} \tag{54}$$

and as we saw in the tutorial, the last line gives zero contribution to the path integral in Eq. (51) since,

$$\int_0^{t_f}\left(m\dot{x}_{\rm cl}\dot{y} - m\omega^2 x_{\rm cl} y + Jy\right) dt = -\int_0^{t_f}\left(m\ddot{x}_{\rm cl} + m\omega^2 x_{\rm cl} - J\right) y\, dt = 0, \tag{55}$$

where the last equality follows the classical EoM (Eq. 53). Therefore, we need to solve the following path integral

$$\begin{aligned} &\int_{x(t=0)=0}^{x(t=t_f)=x_f} \mathcal{D}x(t) \exp\left\{\frac{i}{\hbar}\int_0^{t_f}\left[\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2 + J(t)\,x(t)\right] dt\right\} \\ &\qquad = e^{\frac{i}{\hbar}S_{\rm cl}} \int_{y(t=0)=0}^{y(t=t_f)=0} \mathcal{D}y(t) \exp\left\{\frac{i}{\hbar}\int_0^{t_f}\left[\frac{1}{2}m\dot{y}^2 - \frac{1}{2}m\omega^2 y^2\right] dt\right\} = \sqrt{\frac{m\omega}{2\pi i\hbar \sin(\omega t_f)}}\, e^{\frac{i}{\hbar}S_{\rm cl}}, \end{aligned} \tag{56}$$

where we used the known result for the path integral of simple harmonic oscillator. The classical action is

$$\begin{aligned} S_{\rm cl} \;&=\; \int_0^{t_f}\left[\frac{1}{2}m\dot{x}_{\rm cl}^2 - \frac{1}{2}m\omega^2 x_{\rm cl}^2 + Jx_{\rm cl}\right] dt \overset{(1)}{=} \left[\frac{1}{2}m\dot{x}_{\rm cl}x_{\rm cl}\right]_0^{t_f} - \int_0^{t_f}\left[\frac{1}{2}m\ddot{x}_{\rm cl} + \frac{1}{2}m\omega^2 x_{\rm cl} - J\right] x_{\rm cl} dt \\ &\overset{(2)}{=} \left[\frac{1}{2}m\dot{x}_{\rm cl}x_{\rm cl}\right]_0^{t_f} + \frac{1}{2}\int_0^{t_f} Jx_{\rm cl} dt. \end{aligned} \tag{57}$$

Here, in equality (1) we performed integration by parts, and in equality (2) we used the classical EoM (Eq. 53). The solution to the classical path is given by

$$x_{\rm cl}(t) = x_{\rm hom}(t) + \int_0^{t_f} G(t, t')\, J(t')\, dt', \tag{58}$$

where $x_{\text{hom}}(t)$ is the solution to the homogeneous equation of motion $m\ddot{x}_{\text{hom}} + m\omega^2 x_{\text{hom}} = 0$, which is

$$x_{\text{hom}}(t) = A\cos(\omega t) + B\sin(\omega t)\,. \tag{59}$$

Since $x(t=0)=0$, we know that $A=0$. The other boundary condition, $x(t=t_i)=x_f$, implies

$$x_{\text{cl}}(t) = \frac{\sin(\omega t)}{\sin(\omega t_f)} x_f + \int_0^{t_f} G(t,t')\,J(t')\,dt'. \tag{60}$$

Plugging that result back in Eq. (57) gives the following expression for the action (remember that the Green function vanishes at the endpoints),

$$\boxed{S_{\text{cl}} = \frac{1}{2} m\omega x_f^2 \cot(\omega t_f) + \frac{x_f}{2\sin(\omega t_f)} \int_0^{t_f} J(t)\sin(\omega t)\,dt + \frac{1}{2}\int_0^{t_f}\int_0^{t_f} G(t,t')\,J(t)\,J(t')\,dt dt'}. \tag{61}$$

For the Green function, we need to solve

$$m\left(\frac{d^2}{dt^2} + \omega^2\right) G(t,t') = \delta(t-t')\,. \tag{62}$$

For $t' \neq t$, the solution is a combination of cos and sin. Because the Green function vanishes at the endpoints, we have

$$G(t,t') = \begin{cases} A\sin(\omega t) & 0 \le t \le t' \le t_f \\ B\sin[\omega(t_f - t)] & 0 \le t' \le t \le t_f \end{cases}. \tag{63}$$

From the continuity at $t' = t$,

$$A\sin(\omega t') = B\sin[\omega(t_f - t')] \equiv C\sin(\omega t')\sin[\omega(t_f - t')]\,, \tag{64}$$

thus

$$G(t,t') = C\begin{cases} \sin[\omega(t_f - t')]\sin(\omega t) & 0 \le t \le t' \le t_f \\ \sin[\omega(t_f - t)]\sin(\omega t') & 0 \le t' \le t \le t_f \end{cases}. \tag{65}$$

In order to find the constant C, we integrate Eq. (62) around $t' \pm \epsilon$.

$$m\left(\left.\frac{dG(t,t')}{dt}\right|_{t'+\epsilon} - \left.\frac{dG(t,t')}{dt}\right|_{t'-\epsilon}\right) = 1, \tag{66}$$

or

$$-m\omega C\left[\cos[\omega(t_f - t')]\sin(\omega t') + \sin[\omega(t_f - t')]\cos(\omega t')\right] = 1, \tag{67}$$

and by using the identity $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$ we find

$$C = -\frac{1}{m\omega\sin(\omega t_f)}, \tag{68}$$

so we conclude

$$\boxed{G(t,t') = -\frac{1}{m\omega\sin(\omega t_f)}\begin{cases} \sin[\omega(t_f - t')]\sin(\omega t) & 0 \le t \le t' \le t_f \\ \sin[\omega(t_f - t)]\sin(\omega t') & 0 \le t' \le t \le t_f \end{cases}}. \tag{69}$$

With the expression we obtained for $S_{\rm cl}$ (Eq. 61), we calculate

$$\frac{\delta}{\delta J(t_1)}\mathrm{e}^{\frac{i}{\hbar}S_{\rm cl}}=\frac{i}{\hbar}\frac{\delta S_{\rm cl}}{\delta J(t_1)}\mathrm{e}^{\frac{i}{\hbar}S_{\rm cl}}=\frac{i}{\hbar}\left[\frac{\sin(\omega t_1)}{\sin(\omega t_f)}\frac{x_f}{2}+\int_0^{t_f}G(t,t_1)\,J(t)\,dt\right]\mathrm{e}^{\frac{i}{\hbar}S_{\rm cl}} \tag{70}$$

$$\begin{aligned}\frac{\delta^2}{\delta J(t_1)\,\delta J(t_2)}\mathrm{e}^{\frac{i}{\hbar}S_{\rm cl}} &= \left(\frac{i}{\hbar}\right)^2\left[\frac{\sin(\omega t_1)}{\sin(\omega t_f)}\frac{x_f}{2}+\int_0^{t_f}G(t,t_1)\,J(t)\,dt\right]\left[\frac{\sin(\omega t_2)}{\sin(\omega t_f)}\frac{x_f}{2}+\int_0^{t_f}G(t,t_2)\,J(t)\,dt\right]\mathrm{e}^{\frac{i}{\hbar}S_{\rm cl}}\\ &\quad+\frac{i}{\hbar}G(t_2,t_1)\,\mathrm{e}^{\frac{i}{\hbar}S_{\rm cl}}\end{aligned} \tag{71}$$

Now, combining Eqs. (51), (56) and (71), we finally find

$$\begin{aligned}\langle x_f,t_f|x(t_2)\,x(t_1)\,|x_i=0,t_i=0\rangle &= \sqrt{\frac{m\omega}{2\pi i\hbar\sin(\omega t_f)}}\\ \times\Bigg\{-i\hbar G(t_2,t_1)&+\left[\frac{\sin(\omega t_1)}{\sin(\omega t_f)}\frac{x_f}{2}+\int_0^{t_f}G(t,t_1)\,J(t)\,dt\right]\left[\frac{\sin(\omega t_2)}{\sin(\omega t_f)}\frac{x_f}{2}+\int_0^{t_f}G(t,t_2)\,J(t)\,dt\right]\Bigg\}\mathrm{e}^{\frac{i}{\hbar}S_{\rm cl}}.\end{aligned} \tag{72}$$

Solution to question 6

Solution to item 1

The semi-classical approximation formula which includes also reflection by hard walls,

$$K(x,t;y,0)=\sqrt{\frac{1}{2\pi i\hbar}\left|\frac{\partial^2 S_{\rm cl}}{\partial x\partial y}\right|}\mathrm{e}^{\frac{i}{\hbar}S_{\rm cl}-ir_p\pi}, \tag{73}$$

where r_p is the Maslov index; it corresponds to the number of times the particle hits a wall in its trajectory.

For a free particle that travels a distance d at time t, the classical action is given by

$$S_{\rm cl}=\frac{m}{2t}d^2=\frac{m}{2t}(x-y)^2. \tag{74}$$

Thus, we can identify

$$\sqrt{\frac{1}{2\pi i\hbar}\left|\frac{\partial^2 S_{\rm cl}}{\partial x\partial y}\right|}=\sqrt{\frac{m}{2\pi i\hbar t}}. \tag{75}$$

Next, we need to account all the infinite possible classical trajectories the particle can have as it gets from y to x within time t, resulting a slight modification to Eq. (73),

$$K(x,t;y,0)=\sum_{\text{all classical paths}}\sqrt{\frac{1}{2\pi i\hbar}\left|\frac{\partial^2 S_{\rm cl}}{\partial x\partial y}\right|}\mathrm{e}^{\frac{i}{\hbar}S_{\rm cl}-ir_p\pi}. \tag{76}$$

All the possible trajectories can be classified into 4 groups (where $n=0,1,2,...$) — see Fig. 1:

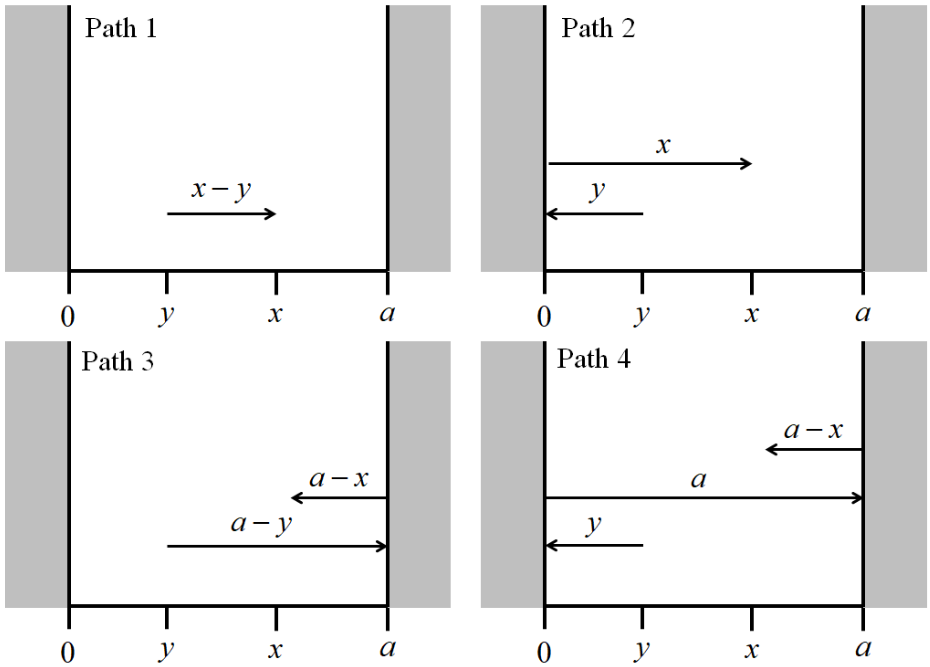


Figure 1: The four fundamental classical paths of a particle in an infinite well that starts at y and finishes at x. In each panel, time flows upwards. Other classical paths can be constructed by taking one of these 4 paths, and letting the particle travel an additional distance of $2na$. Notice that the particle hits the wall even number of times in paths 1 and 4, but it hits the wall only odd number of times in paths 2 and 3.

1. $d_1(n) = x - y + 2na$
2. $d_2(n) = y + x + 2na$
3. $d_3(n) = (a - y) + (a - x) + 2na = 2(n+1)a - x - y$
4. $d_4(n) = y + a + (a - x) + 2na = 2(n+1)a + y - x$

Notice that regardless of the exact trajectory the particle has chosen, $|\partial^2 S_{\mathrm{cl}}/\partial x \partial y| = m/t$, and so we are only required to calculate

$$\sum_{\text{all classical paths}} \mathrm{e}^{iS_{\mathrm{cl}}/\hbar - ir_p\pi} = \sum_{n=0}^{\infty}\left[+\mathrm{e}^{\frac{i}{\hbar}\frac{m}{2t}d_1^2(n)} - \mathrm{e}^{\frac{i}{\hbar}\frac{m}{2t}d_2^2(n)} - \mathrm{e}^{\frac{i}{\hbar}\frac{m}{2t}d_3^2(n)} + \mathrm{e}^{\frac{i}{\hbar}\frac{m}{2t}d_4^2(n)}\right]. \tag{77}$$

Notice that due to Maslov index, paths of type 1 and 4 are assigned with a plus sign while paths of type 2 and 3 are assigned with a minus sign. This is because in the former paths the particle hits the walls even number of times, while in the latter paths it hits the walls odd number of times.

Thus, we have

$$
\begin{aligned}
\sum_{\text{all classical paths}} \mathrm{e}^{iS_{\mathrm{cl}}/\hbar - ir_p\pi} &= \sum_{n=0}^{\infty} \exp\left[\frac{i}{\hbar}\frac{m}{2t}(x-y+2na)^2\right] + \sum_{n=0}^{\infty} \exp\left[\frac{i}{\hbar}\frac{m}{2t}(2(n+1)a+y-x)^2\right] \\
&- \sum_{n=0}^{\infty} \exp\left[\frac{i}{\hbar}\frac{m}{2t}(y+x+2na)^2\right] - \sum_{n=0}^{\infty} \exp\left[\frac{i}{\hbar}\frac{m}{2t}(2(n+1)a-x-y)^2\right] \\
&= \sum_{n=0}^{\infty} \exp\left[\frac{i}{\hbar}\frac{m}{2t}(x-y+2na)^2\right] + \sum_{n=1}^{\infty} \exp\left[\frac{i}{\hbar}\frac{m}{2t}(x-y-2na)^2\right] \\
&- \sum_{n=0}^{\infty} \exp\left[\frac{i}{\hbar}\frac{m}{2t}(x+y+2na)^2\right] - \sum_{n=1}^{\infty} \exp\left[\frac{i}{\hbar}\frac{m}{2t}(x+y-2na)^2\right] \\
&= \sum_{n=-\infty}^{\infty} \left\{\exp\left[\frac{im(x-y+2na)^2}{2\hbar t}\right] - \exp\left[\frac{im(x+y+2na)^2}{2\hbar t}\right]\right\}. \qquad (78)
\end{aligned}
$$

Combinning Eqs. (76) and (78), we conclude,

$$
\boxed{K(x,t;y,0) = \sqrt{\frac{m}{2\pi i\hbar t}} \sum_{n=-\infty}^{\infty} \left\{\exp\left[\frac{im(x-y+2na)^2}{2\hbar t}\right] - \exp\left[\frac{im(x+y+2na)^2}{2\hbar t}\right]\right\}}. \qquad (79)
$$

Solution to item 2

Recall the formula from recitation 5,

$$
\begin{aligned}
\sum_n \mathrm{e}^{-iE_nt/\hbar} &= \int_0^a dx K(x,t;x,0) \\
&= \sqrt{\frac{m}{2\pi i\hbar t}} \int_0^a dx \sum_{n=-\infty}^{\infty} \left\{\exp\left[\frac{im(2na)^2}{2\hbar t}\right] - \exp\left[\frac{im(2x+2na)^2}{2\hbar t}\right]\right\}. \qquad (80)
\end{aligned}
$$

The second term is easier to calculate

$$
\begin{aligned}
\sqrt{\frac{m}{2\pi i\hbar t}} \sum_{n=-\infty}^{\infty} \int_0^a dx \exp\left[\frac{im(2x+2na)^2}{2\hbar t}\right] &\overset{y=x+na}{=} \sqrt{\frac{m}{2\pi i\hbar t}} \sum_{n=-\infty}^{\infty} \int_{na}^{(n+1)a} dy \exp\left(i\frac{2m}{\hbar t}y^2\right) \\
&= \sqrt{\frac{m}{2\pi i\hbar t}} \int_{-\infty}^{\infty} dy \exp\left(i\frac{2m}{\hbar t}y^2\right) = \sqrt{\frac{m}{2\pi i\hbar t}}\sqrt{\frac{\pi i\hbar t}{2m}} = \frac{1}{2}. \qquad (81)
\end{aligned}
$$

For the first term, the integration is trivial.

$$
\begin{aligned}
\sqrt{\frac{m}{2\pi i\hbar t}} \int_0^a dx \sum_{n=-\infty}^{\infty} \exp\left[\frac{im(2na)^2}{2\hbar t}\right] &= \sqrt{\frac{ma^2}{2\pi i\hbar t}} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{2ma^2n^2}{i\hbar t}\right) \\
&= \alpha \sum_{n=-\infty}^{\infty} \mathrm{e}^{-4\pi\alpha^2n^2} = \alpha \sum_{n=-\infty}^{\infty} f(\alpha n), \qquad (82)
\end{aligned}
$$

where we have defined $\alpha \equiv (ma^2/2\pi i\hbar t)^{1/2}$ and $f(x) \equiv \mathrm{e}^{-4\pi x^2}$. We now compute

$$
F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{ikx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-4\pi x^2+ikx} dx = \frac{1}{2\sqrt{2\pi}} \mathrm{e}^{-k^2/16\pi}, \qquad (83)
$$

where we have used $\int_{-\infty}^{\infty} \mathrm{e}^{-ax^2+bx} = \sqrt{\frac{\pi}{a}} \mathrm{e}^{b^2/4a}$.

We can now use Eq. (82) and the Poisson summation formula to write the first term in Eq. (80) as

$$\begin{aligned}
\sqrt{\frac{m}{2\pi i\hbar t}} \int_0^a dx \sum_{n=-\infty}^{\infty} \exp\left[\frac{im\,(2na)^2}{2\hbar t}\right] &= \sqrt{2\pi} \sum_{n=-\infty}^{\infty} F\left(\frac{2\pi n}{\alpha}\right) \\
&= \frac{1}{2} \sum_{n=-\infty}^{\infty} \mathrm{e}^{-\pi n^2/4\alpha^2} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \mathrm{e}^{-i\frac{\pi^2\hbar^2 n^2}{2ma^2} t/\hbar}. \qquad (84)
\end{aligned}$$

Combining Eqs. (80), (81) and (84), we get

$$\sum_n \mathrm{e}^{-iE_n t/\hbar} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \mathrm{e}^{-i\frac{\pi^2\hbar^2 n^2}{2ma^2} t/\hbar} - \frac{1}{2} = \sum_{n=1}^{\infty} \mathrm{e}^{-i\frac{\pi^2\hbar^2 n^2}{2ma^2} t/\hbar}, \qquad (85)$$

and by comparing the LHS with the RHS we can retrieve the energy levels:

$$\boxed{E_n = \frac{\pi^2\hbar^2 n^2}{2ma^2}}. \qquad (86)$$

For those of you who are interested, below is the proof for Poisson summation formula. Let us begin from the RHS and then plug the definition for $F(k)$.

$$\begin{aligned}
\frac{\sqrt{2\pi}}{\alpha} \sum_{n=-\infty}^{\infty} F\left(\frac{2\pi n}{\alpha}\right) &= \frac{\sqrt{2\pi}}{\alpha} \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{i2\pi n x/\alpha} dx = \int_{-\infty}^{\infty} dy f(y\alpha) \sum_{n=-\infty}^{\infty} \mathrm{e}^{i2\pi n y} \\
&\overset{(*)}{=} \int_{-\infty}^{\infty} dy f(y\alpha) \sum_{n=-\infty}^{\infty} \delta(y-n) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dy f(y\alpha)\, \delta(y-n) \\
&= \sum_{n=-\infty}^{\infty} f(n\alpha). \qquad (87)
\end{aligned}$$

The equality (*) is due to the famous identity

$$\sum_{n=-\infty}^{\infty} \delta(y-n) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{i2\pi n y}, \qquad (88)$$

which can be derived via the discrete Fourier expansion of the LHS (all the Fourier coefficients are simply 1).