Quantum Mechanics 3 - Homework 5 Solution

Solution to question 1

Solution to item a

We expand $\Lambda^{\mu}{}_{\nu}$ for a small ϵ parameter.

$$\Lambda^{\mu}{}_{\nu} = \begin{bmatrix} \cosh\epsilon & 0 & 0 & -\sinh\epsilon \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\epsilon & 0 & 0 & \cosh\epsilon \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -\epsilon \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\epsilon & 0 & 0 & 1 \end{bmatrix} + \mathcal{O}\left(\epsilon^2\right) = \delta^{\mu}_{\nu} + \epsilon^{\mu}{}_{\nu} + \mathcal{O}\left(\epsilon^2\right), \tag{1}$$

where we identified

$$\epsilon^{\mu}{}_{\nu} = \begin{bmatrix} 0 & 0 & 0 & -\epsilon \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\epsilon & 0 & 0 & 0 \end{bmatrix}. \tag{2}$$

Now, we compute

$$\epsilon^{\mu\rho} = \epsilon^{\mu}{}_{\nu}\eta^{\nu\rho} = \begin{bmatrix} 0 & 0 & 0 & -\epsilon \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\epsilon & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & +\epsilon \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\epsilon & 0 & 0 & 0 \end{bmatrix}, \tag{3}$$

which is clearly an anti-symmetric matrix.

Solution to item b

For rotations around the z-axis by an angle θ, the matrix transformation is

$$\Lambda^{\mu}{}_{\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4}$$

Expanding the matrix for a small ϵ parameter yields

$$\Lambda^{\mu}{}_{\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\epsilon & 0 \\ 0 & \epsilon & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \delta^{\mu}_{\nu} + \epsilon^{\mu}{}_{\nu} + \mathcal{O}\left(\epsilon^2\right), \tag{5}$$

where now

$$\epsilon^{\mu}{}_{\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\epsilon & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{6}$$

and therefore

$$\epsilon^{\mu\rho} = \epsilon^{\mu}{}_{\nu}\eta^{\nu\rho} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\epsilon & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & -\epsilon & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{7}$$

and we see that it is also an anti-symmetric matrix.

Solution to item c

We know that 4-vectors undergo Lorentz transformations according to

$$x^{\mu} \to x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu} = \Lambda^{\mu}{}_{\nu}\eta^{\nu\rho}x_{\rho} \equiv \Lambda^{\mu\rho}x_{\rho}, \tag{8}$$

and thus

$$x^{\mu}x_{\mu} = \eta_{\mu\nu}x^{\mu}x^{\nu} \to \eta_{\mu\nu}x'^{\mu}x'^{\nu} = \eta_{\mu\nu}\Lambda^{\mu\rho}x_{\rho}\Lambda^{\nu\sigma}x_{\sigma} = \left(\eta_{\mu\nu}\Lambda^{\mu\rho}\Lambda^{\nu\sigma}\right)x_{\rho}x_{\sigma}. \tag{9}$$

Since Lorentz transformations preserve the norm of 4-vectors, Eq (9) implies that

$$\eta^{\rho\sigma} = \eta_{\mu\nu}\Lambda^{\mu\rho}\Lambda^{\nu\sigma}. \tag{10}$$

We now consider infinitesimal Lorentz transformations,

$$\begin{aligned} \eta^{\rho\sigma} &= \eta_{\mu\nu}\Lambda^{\mu\rho}\Lambda^{\nu\sigma} = \eta_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\eta^{\alpha\rho}\Lambda^{\nu}{}_{\beta}\eta^{\beta\sigma} = \eta_{\mu\nu}\eta^{\alpha\rho}\eta^{\beta\sigma}\left(\delta^{\mu}_{\alpha} + \epsilon^{\mu}{}_{\alpha}\right)\left(\delta^{\nu}_{\beta} + \epsilon^{\nu}{}_{\beta}\right) \\ &= \eta_{\mu\nu}\left(\eta^{\mu\rho} + \epsilon^{\mu}{}_{\alpha}\eta^{\alpha\rho}\right)\left(\eta^{\nu\sigma} + \epsilon^{\nu}{}_{\beta}\eta^{\beta\sigma}\right) = \eta_{\mu\nu}\left(\eta^{\mu\rho} + \epsilon^{\mu\rho}\right)\left(\eta^{\nu\sigma} + \epsilon^{\nu\sigma}\right) \\ &= \eta_{\mu\nu}\eta^{\mu\rho}\eta^{\nu\sigma} + \eta_{\mu\nu}\eta^{\nu\sigma}\epsilon^{\mu\rho} + \eta_{\mu\nu}\eta^{\mu\rho}\epsilon^{\nu\sigma} + \mathcal{O}\left(\epsilon^{2}\right) \\ &= \eta^{\rho\sigma} + \delta^{\sigma}_{\mu}\epsilon^{\mu\rho} + \delta^{\rho}_{\nu}\epsilon^{\nu\sigma} + \mathcal{O}\left(\epsilon^{2}\right) = \eta^{\rho\sigma} + \epsilon^{\sigma\rho} + \epsilon^{\rho\sigma} + \mathcal{O}\left(\epsilon^{2}\right), \end{aligned} \tag{11}$$

therefore, in order the LHS of Eq. (11) be equal to the RHS, $\epsilon^{\rho\sigma}$ must be an anti-symmetric matrix,

$$\boxed{\epsilon^{\sigma\rho} = -\epsilon^{\rho\sigma}}. \tag{12}$$

Solution to question 2

Solution to item a

First, we write the EM tensor in both covariant and contra-variant forms

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E^x & -E^y & -E^z \\ E^x & 0 & -B^z & B^y \\ E^y & B^z & 0 & -B^x \\ E^z & -B^y & B^x & 0 \end{bmatrix} \implies F_{\mu\nu} = \eta_{\mu\rho}\eta_{\nu\sigma}F^{\rho\sigma} = \begin{bmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & -B^z & B^y \\ -E^y & B^z & 0 & -B^x \\ -E^z & -B^y & B^x & 0 \end{bmatrix}. \tag{13}$$

Thus, the Lagrangian of the free EM theory is

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = -\frac{1}{4}\sum_{\mu,\nu=0}^{3} F^{\mu\nu}F_{\mu\nu} = -\frac{1}{4}2\left(\vec{B}^2 - \vec{E}^2\right) = \frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right). \tag{14}$$

We now need to evaluate the energy-momentum tensor,

$$T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial\left(\partial^\mu A^\lambda\right)}\partial_\nu A^\lambda - \eta_{\mu\nu}\mathcal{L} = -F_{\mu\lambda}\partial_\nu A^\lambda - \eta_{\mu\nu}\mathcal{L}. \tag{15}$$

The 00 component is

$$\begin{aligned} T^{00} &= -F^{0\lambda}\partial^0 A_\lambda - \eta^{00}\mathcal{L} = -F^{0i}\partial^0 A_i - \mathcal{L} = E^i\partial^0 A_i + \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \\ &= = -E^i\partial^0 A^i + \frac{1}{2}\left(\vec{B}^2 - \vec{E}^2\right) \end{aligned} \tag{16}$$

and the Hamiltonian is

$$H = \int d^3x T^{00} = -\int d^3x E^i \partial^0 A^i + \frac{1}{2}\int d^3x \left(\vec{B}^2 - \vec{E}^2\right). \tag{17}$$

In order to write that Hamiltonian in terms of the $\vec{E}$ and $\vec{B}$ fields, we "complete" the first term to F^{0i} and then perform integration by parts

$$\begin{aligned} -\int d^3x E^i\partial^0 A^i &= -\int d^3x E^i\left(\partial^0 A^i - \partial^i A^0\right) - \int d^3x E^i \partial^i A^0 = -\int d^3x E^i F^{0i} + \int d^3x A^0 \partial^i E^i \\ &= \int d^3x E^i E^i - \int d^3x A^0 \partial_i E^i = \int d^3x \vec{E}^2 - \int d^3x \phi \left(\vec{\nabla}\cdot\vec{E}\right) = \int d^3x \vec{E}^2, \end{aligned} \tag{18}$$

where the last equality is due to Maxwell equation in vacuum ($\vec{\nabla}\cdot\vec{E} = 0$). Thus

$$\boxed{H = \frac{1}{2}\int d^3x \left(\vec{E}^2 + \vec{B}^2\right)}. \tag{19}$$

Solution to item b

The $0i$ component of $T^{\mu\nu}$ is

$$T^{0i} = -F^{0\lambda}\partial^i A_\lambda = -F^{0n}\partial^i A_n = E^n \partial^i A_n = -E^n\left(\partial^i A^n\right). \tag{20}$$

Very similarly to the calculations made in Eq. (18), we find

$$\begin{aligned}
-\int d^3x E^n \partial^i A^n &= -\int d^3x E^n \left(\partial^i A^n - \partial^n A^i\right) - \int d^3x E^n \partial^n A^i = -\int d^3x E^n F^{in} + \int d^3x A^i \partial^n E^n \\
&= \int d^3x E^n \epsilon^{ink} B^k - \int d^3x A^i \partial_n E^n = \int d^3x \left(\vec{E}\times\vec{B}\right)^i - \int d^3x A^i \left(\vec{\nabla}\cdot\vec{E}\right) \\
&= \int d^3x \left(\vec{E}\times\vec{B}\right)^i, \qquad (21)
\end{aligned}$$

and thus

$$\boxed{\vec{p} = \int d^3x \left(\vec{E}\times\vec{B}\right)}. \qquad (22)$$

Solution to item c

The angular momentum tensor of the EM field is given by

$$J_{\mu\alpha\beta} = \left(T_{\mu\alpha}x_\beta - T_{\mu\beta}x_\alpha\right) + \frac{\partial\mathcal{L}}{\partial\left(\partial^\mu A^\nu\right)}\frac{\partial\left(\delta^{(0)}A^\nu\right)}{\partial\epsilon^{\alpha\beta}} = \left(T_{\mu\alpha}x_\beta - T_{\mu\beta}x_\alpha\right) - F_{\mu\nu}\frac{\partial\left(\delta^{(0)}A^\nu\right)}{\partial\epsilon^{\alpha\beta}}. \qquad (23)$$

Since for a vector field $\delta^{(0)}A^\nu = \epsilon^{\nu\rho}A_\rho$, and $\epsilon^{\nu\rho}$ is anti-symmetric, we find that

$$J_{\mu\alpha\beta} = \left(T_{\mu\alpha}x_\beta - T_{\mu\beta}x_\alpha\right) - \left(F_{\mu\alpha}A_\beta - F_{\mu\beta}A_\alpha\right). \qquad (24)$$

The Noether charges associated with the above tensor are

$$M^{jk} = \int d^3x \left[\left(T^{0j}x^k - T^{0k}x^j\right) - \left(F^{0j}A^k - F^{0k}A^j\right)\right], \qquad (25)$$

and the components of the angular momentum are given by

$$\begin{aligned}
J^i &= \frac{1}{2}\epsilon^{ijk}M^{jk} = \int d^3x \left(\epsilon^{ijk}T^{0j}x^k - \epsilon^{ijk}F^{0j}A^k\right) = \int d^3x \left(-\epsilon^{ijk}E^n\left(\partial^j A^n\right)x^k - \epsilon^{ijk}F^{0j}A^k\right) \\
&= \int d^3x \left[E^n\left(\epsilon^{ikj}x^k\partial^j\right)A^n + \epsilon^{ijk}E^j A^k\right], \qquad (26)
\end{aligned}$$

and so we conclude

$$\boxed{\vec{J} = \sum_{n=x,y,z}\int d^3x \left[E^n\left(\vec{x}\times\vec{\nabla}\right)A^n\right] + \int d^3x\left(\vec{E}\times\vec{A}\right)}. \qquad (27)$$

The first term in Eq. (27) is $\vec{L}$ while the second one is $\vec{S}$. None of these terms are gauge invariant as they depend explicitly on $\vec{A}$.

Solution to item d

We inspect the i'th component of $\int d^3x \left[\vec{x} \times \left(\vec{E} \times \vec{B}\right)\right]$:

$$\begin{aligned}
\int d^3x \left[\vec{x} \times \left(\vec{E} \times \vec{B}\right)\right]^i &= \int d^3x \left[\vec{x} \times \left(\vec{E} \times \left(\vec{\nabla} \times \vec{A}\right)\right)\right]^i = \int d^3x \epsilon^{ijk} x^j \epsilon^{klm} E^l \epsilon^{mab} \left(\partial_a A^b\right) \\
&= \int d^3x \epsilon^{ijk} \epsilon^{klm} \epsilon^{mab} x^j E^l \left(\partial_a A^b\right) = \int d^3x \epsilon^{ijk} \left(\delta^{ka}\delta^{lb} - \delta^{kb}\delta^{la}\right) x^j E^l \left(\partial_a A^b\right) \\
&= \int d^3x \epsilon^{ijk} \left(x^j E^l \left(\partial_k A^l\right) - x^j E^l \left(\partial_l A^k\right)\right) \\
&= \int d^3x E^l \left(\epsilon^{ijk} x^j \partial_k\right) A^l + \int d^3x \epsilon^{ijk} A^k \partial_l \left(x^j E^l\right) \\
&= \int d^3x E^l \left(\vec{x} \times \vec{\nabla}\right)^i A^l + \int d^3x \epsilon^{ijk} A^k \left[\left(\partial_l x^j\right) E^l + \left(\partial_l E^l\right) x^j\right] \\
&\overset{(*)}{=} \int d^3x E^l \left(\vec{x} \times \vec{\nabla}\right)^i A^l + \int d^3x \epsilon^{ijk} A^k \delta^{lj} E^l \\
&= \int d^3x E^l \left(\vec{x} \times \vec{\nabla}\right)^i A^l + \int d^3x \epsilon^{ijk} E^j A^k \\
&= \int d^3x E^l \left(\vec{x} \times \vec{\nabla}\right)^i A^l + \int d^3x \left(\vec{E} \times \vec{A}\right)^i, \qquad (28)
\end{aligned}$$

where equality (*) is due to Gauss law ($\partial_l E^l = 0$). Therefore, by combining Eq. (27)-(28) we see that

$$\boxed{\vec{J} = \int d^3x \left[\vec{x} \times \left(\vec{E} \times \vec{B}\right)\right]}. \qquad (29)$$

This expression is manifestly gauge invariant.

Solution to question 3

The energy-momentum tensor for the scalar field is

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi\right)} \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L} = \partial^\mu \phi \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L}. \qquad (30)$$

The $0i$ component is

$$T^{0i} = \partial^0 \phi \partial^i \phi = \pi \partial^i \phi, \qquad (31)$$

and therefore the (quantized) momentum of the scalar field is

$$\hat{P}^i = \int d^3x \, \hat{\pi}(x) \, \partial^i \hat{\phi}(x) = -\int d^3x \, \hat{\pi}(x) \, \partial_i \hat{\phi}(x). \qquad (32)$$

For the real scalar field we have

$$\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(\hat{a}_{\vec{p}} \, \mathrm{e}^{-ip\cdot x} + \hat{a}^\dagger_{\vec{p}} \, \mathrm{e}^{+ip\cdot x}\right) \qquad (33)$$

$$\hat{\pi}(x) = -i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{E_p}{2}} \left(\hat{a}_{\vec{p}} \, \mathrm{e}^{-ip\cdot x} - \hat{a}^\dagger_{\vec{p}} \, \mathrm{e}^{+ip\cdot x}\right), \qquad (34)$$

and thus

$$\partial_i \hat{\phi}(x) = i \int \frac{d^3p}{(2\pi)^3} \frac{p^i}{\sqrt{2E_p}} \left(\hat{a}_{\vec{p}} \, \mathrm{e}^{-ip\cdot x} - \hat{a}^\dagger_{\vec{p}} \, \mathrm{e}^{+ip\cdot x} \right). \tag{35}$$

So we now need to compute

$$\begin{aligned}
\hat{P}^i &= -\int d^3x \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{E_p}{2}} \left(\hat{a}_{\vec{p}} \, \mathrm{e}^{-ip\cdot x} - \hat{a}^\dagger_{\vec{p}} \, \mathrm{e}^{+ip\cdot x} \right) \int \frac{d^3p'}{(2\pi)^3} \frac{p'^i}{\sqrt{2E_{p'}}} \left(\hat{a}_{\vec{p}'} \, \mathrm{e}^{-ip'\cdot x} - \hat{a}^\dagger_{\vec{p}'} \, \mathrm{e}^{+ip'\cdot x} \right) \\
&= -\frac{1}{2} \iint \frac{d^3p d^3p'}{(2\pi)^6} \sqrt{\frac{E_p}{E_{p'}}} p'^i \int d^3x \begin{pmatrix} +\hat{a}_{\vec{p}} \, \hat{a}_{\vec{p}'} \, \mathrm{e}^{-i(p+p')\cdot x} + \hat{a}^\dagger_{\vec{p}} \, \hat{a}^\dagger_{\vec{p}'} \, \mathrm{e}^{+i(p+p')\cdot x} \\ -\hat{a}_{\vec{p}} \, \hat{a}^\dagger_{\vec{p}'} \, \mathrm{e}^{-i(p-p')\cdot x} - \hat{a}^\dagger_{\vec{p}} \, \hat{a}_{\vec{p}'} \, \mathrm{e}^{+i(p-p')\cdot x} \end{pmatrix} \\
&= -\frac{1}{2} \iint \frac{d^3p d^3p'}{(2\pi)^3} \sqrt{\frac{E_p}{E_{p'}}} p'^i \begin{bmatrix} +\hat{a}_{\vec{p}} \, \hat{a}_{\vec{p}'} \, \mathrm{e}^{-i\left(E_p+E_{p'}\right)t} \delta^{(3)}\left(\vec{p}+\vec{p}'\right) + \hat{a}^\dagger_{\vec{p}} \, \hat{a}^\dagger_{\vec{p}'} \, \mathrm{e}^{+i\left(E_p+E_{p'}\right)t} \delta^{(3)}\left(\vec{p}+\vec{p}'\right) \\ -\hat{a}_{\vec{p}} \, \hat{a}^\dagger_{\vec{p}'} \, \mathrm{e}^{-i\left(E_p-E_{p'}\right)t} \delta^{(3)}\left(\vec{p}-\vec{p}'\right) - \hat{a}^\dagger_{\vec{p}} \, \hat{a}_{\vec{p}'} \, \mathrm{e}^{+i\left(E_p-E_{p'}\right)t} \delta^{(3)}\left(\vec{p}-\vec{p}'\right) \end{bmatrix} \\
&= -\frac{1}{2} \int d^3p \, p^i \left(\hat{a}_{\vec{p}} \, \hat{a}_{-\vec{p}} \, \mathrm{e}^{-2iE_pt} + \hat{a}^\dagger_{\vec{p}} \, \hat{a}^\dagger_{-\vec{p}} \, \mathrm{e}^{+2iE_pt} \right) + \frac{1}{2} \int d^3p \, p^i \left(\hat{a}_{\vec{p}} \, \hat{a}^\dagger_{\vec{p}} + \hat{a}^\dagger_{\vec{p}} \, \hat{a}_{\vec{p}} \right) \\
&\overset{(1)}{=} \frac{1}{2} \int d^3p \, p^i \left(\hat{a}_{\vec{p}} \, \hat{a}^\dagger_{\vec{p}} + \hat{a}^\dagger_{\vec{p}} \, \hat{a}_{\vec{p}} \right) \overset{(2)}{=} \int d^3p \, p^i \hat{a}^\dagger_{\vec{p}} \, \hat{a}_{\vec{p}} + \frac{(2\pi)^3 \, \delta^{(3)}(0)}{2} \int d^3p \, p^i.
\end{aligned} \tag{36}$$

The justification for equality (1) is because $p^i \hat{a}_{\vec{p}} \, \hat{a}_{-\vec{p}}$ is anti-symmetric under parity, while the integration is performed over the entire p-space. Equality (2) is due to $\left[\hat{a}_{\vec{p}} \, , \hat{a}^\dagger_{\vec{p}'} \right] = (2\pi)^3 \, \delta^{(3)}\left(\vec{p}-\vec{p}'\right)$. Since $\int d^3p \, p^i = 0$ we conclude (by restoring back $\hbar$ and changing the integration variable to k)

$$\boxed{\hat{\vec{P}} = \int d^3k \, \hbar\vec{k} \, \hat{a}^\dagger_{\vec{k}} \, \hat{a}_{\vec{k}}}. \tag{37}$$

This is the result we should have expected. $\hat{a}^\dagger_{\vec{k}} \, \hat{a}_{\vec{k}}$ is the number density (in k-space) of particles with momentum $\hbar\vec{k}$, and we integrate over all the possible values of $\vec{k}$.

Solution to question 4

The quantized momentum of the EM field

$$\hat{\vec{P}} = \frac{1}{2c} \int d^3x \left(\hat{\vec{E}} \times \hat{\vec{B}} - \hat{\vec{B}} \times \hat{\vec{E}} \right). \tag{38}$$

The EM fields, in terms of creation and annihilation operators,

$$\hat{\vec{E}}\left(\vec{r},t\right) = i \int \frac{d^3k}{\sqrt{(2\pi)^3}} \sqrt{\frac{\hbar\omega_k}{2}} \sum_{\lambda=1,2} \left[\mathrm{e}^{i\left(\vec{k}\cdot\vec{r}-\omega_k t\right)} \vec{\epsilon}_{\vec{k},\lambda} \hat{a}_{\vec{k},\lambda} - \mathrm{e}^{-i\left(\vec{k}\cdot\vec{r}-\omega_k t\right)} \vec{\epsilon}^{\,*}_{\vec{k},\lambda} \hat{a}^\dagger_{\vec{k},\lambda} \right] \tag{39}$$

$$\hat{\vec{B}}\left(\vec{r},t\right) = i \int \frac{d^3k}{\sqrt{(2\pi)^3}} \sqrt{\frac{\hbar\omega_k}{2}} \sum_{\lambda=1,2} \hat{k} \times \left[\mathrm{e}^{i\left(\vec{k}\cdot\vec{r}-\omega_k t\right)} \vec{\epsilon}_{\vec{k},\lambda} \hat{a}_{\vec{k},\lambda} - \mathrm{e}^{-i\left(\vec{k}\cdot\vec{r}-\omega_k t\right)} \vec{\epsilon}^{\,*}_{\vec{k},\lambda} \hat{a}^\dagger_{\vec{k},\lambda} \right]. \tag{40}$$

We shall use index notation to calculate the cross product.

$$\begin{aligned}
\left(\hat{\vec{E}} \times \hat{\vec{B}} \right)^n &= \epsilon^{nml} \hat{E}^m \hat{B}^l = \epsilon^{nml} i^2 \int \int \frac{d^3k}{\sqrt{(2\pi)^3}} \frac{d^3k'}{\sqrt{(2\pi)^3}} \sqrt{\frac{\hbar\omega_k}{2}} \sqrt{\frac{\hbar\omega_{k'}}{2}} \epsilon^{lab} \hat{k'}^a \\
\times \sum_{\lambda,\lambda'=1,2} & \left[\mathrm{e}^{i\left(\vec{k}\cdot\vec{r}-\omega_k t\right)} \epsilon^m_{\vec{k},\lambda} \hat{a}_{\vec{k},\lambda} - \mathrm{e}^{-i\left(\vec{k}\cdot\vec{r}-\omega_k t\right)} \epsilon^{m*}_{\vec{k},\lambda} \hat{a}^\dagger_{\vec{k},\lambda} \right] \left[\mathrm{e}^{i\left(\vec{k}'\cdot\vec{r}-\omega_{k'} t\right)} \epsilon^b_{\vec{k}',\lambda'} \hat{a}_{\vec{k}',\lambda'} - \mathrm{e}^{-i\left(\vec{k}'\cdot\vec{r}-\omega_{k'} t\right)} \epsilon^{b*}_{\vec{k}',\lambda'} \hat{a}^\dagger_{\vec{k}',\lambda'} \right].
\end{aligned} \tag{41}$$

The last line can be written as

$$+\mathrm{e}^{i\left[\left(\vec{k}+\vec{k}'\right)\cdot\vec{r}-(\omega_k+\omega_{k'})t\right]}\epsilon^m_{\vec{k},\lambda}\epsilon^b_{\vec{k}',\lambda'}\hat{a}_{\vec{k},\lambda}\hat{a}_{\vec{k}',\lambda'}-\mathrm{e}^{-i\left[\left(\vec{k}-\vec{k}'\right)\cdot\vec{r}-(\omega_k-\omega_{k'})t\right]}\epsilon^{m*}_{\vec{k},\lambda}\epsilon^b_{\vec{k}',\lambda'}\hat{a}^\dagger_{\vec{k},\lambda}\hat{a}_{\vec{k}',\lambda'}$$
$$-\mathrm{e}^{i\left[\left(\vec{k}-\vec{k}'\right)\cdot\vec{r}-(\omega_k-\omega_{k'})t\right]}\epsilon^m_{\vec{k},\lambda}\epsilon^{b*}_{\vec{k}',\lambda'}\hat{a}_{\vec{k},\lambda}\hat{a}^\dagger_{\vec{k}',\lambda'}+\mathrm{e}^{-i\left[\left(\vec{k}+\vec{k}'\right)\cdot\vec{r}-(\omega_k+\omega_{k'})t\right]}\epsilon^{m*}_{\vec{k},\lambda}\epsilon^{b*}_{\vec{k}',\lambda'}\hat{a}^\dagger_{\vec{k},\lambda}\hat{a}^\dagger_{\vec{k}',\lambda'}. \tag{42}$$

By performing the d^3x integral (Eq. 38), we would get $\int \mathrm{e}^{i\left(\vec{k}\pm\vec{k}'\right)\cdot\vec{r}} = (2\pi)^3\,\delta^3\left(\vec{k}\pm\vec{k}'\right)$, and then, by performing the d^3k' integral (Eq. 41), we have (note that $\omega_{-\vec{k}}=\omega_{\vec{k}}=\omega_k$)

$$\int d^3x\left(\hat{\vec{E}}\times\hat{\vec{B}}\right)^n = \epsilon^{lnm}\epsilon^{lab}\int d^3k\frac{\hbar\omega_k}{2}\hat{k}^a\sum_{\lambda,\lambda'=1,2}\Big(\mathrm{e}^{-2i\omega_k t}\epsilon^m_{\vec{k},\lambda}\epsilon^b_{-\vec{k},\lambda'}\hat{a}_{\vec{k},\lambda}\hat{a}_{-\vec{k},\lambda'}+\epsilon^{m*}_{\vec{k},\lambda}\epsilon^b_{\vec{k},\lambda'}\hat{a}^\dagger_{\vec{k},\lambda}\hat{a}_{\vec{k},\lambda'}$$
$$+\epsilon^m_{\vec{k},\lambda}\epsilon^{b*}_{\vec{k},\lambda'}\hat{a}_{\vec{k},\lambda}\hat{a}^\dagger_{\vec{k},\lambda'}+\mathrm{e}^{2i\omega_k t}\epsilon^{m*}_{\vec{k},\lambda}\epsilon^{b*}_{-\vec{k},\lambda'}\hat{a}^\dagger_{\vec{k},\lambda}\hat{a}^\dagger_{-\vec{k},\lambda'}\Big). \tag{43}$$

We now use $\epsilon^{lnm}\epsilon^{lab}=\delta^{na}\delta^{mb}-\delta^{nb}\delta^{ma}$. Since $\hat{k}^m\epsilon^m_{\vec{k},\lambda}=\hat{k}\cdot\vec{\epsilon}_{\vec{k},\lambda}=0$, we have

$$\int d^3x\left(\hat{\vec{E}}\times\hat{\vec{B}}\right)^n = \int d^3k\frac{\hbar\omega_k}{2}\hat{k}^n\sum_{\lambda,\lambda'=1,2}\Big(\mathrm{e}^{-2i\omega_k t}\epsilon^m_{\vec{k},\lambda}\epsilon^m_{-\vec{k},\lambda'}\hat{a}_{\vec{k},\lambda}\hat{a}_{-\vec{k},\lambda'}+\epsilon^{m*}_{\vec{k},\lambda}\epsilon^m_{\vec{k},\lambda'}\hat{a}^\dagger_{\vec{k},\lambda}\hat{a}_{\vec{k},\lambda'}$$
$$+\epsilon^m_{\vec{k},\lambda}\epsilon^{m*}_{\vec{k},\lambda'}\hat{a}_{\vec{k},\lambda}\hat{a}^\dagger_{\vec{k},\lambda'}+\mathrm{e}^{2i\omega_k t}\epsilon^{m*}_{\vec{k},\lambda}\epsilon^{m*}_{-\vec{k},\lambda'}\hat{a}^\dagger_{\vec{k},\lambda}\hat{a}^\dagger_{-\vec{k},\lambda'}\Big). \tag{44}$$

Very similarly, we find,

$$\int d^3x\left(\hat{\vec{B}}\times\hat{\vec{E}}\right)^n = \int d^3k\frac{\hbar\omega_k}{2}\hat{k}^n\sum_{\lambda,\lambda'=1,2}\Big(\mathrm{e}^{-2i\omega_k t}\epsilon^l_{\vec{k},\lambda}\epsilon^l_{-\vec{k},\lambda'}\hat{a}_{\vec{k},\lambda}\hat{a}_{-\vec{k},\lambda'}-\epsilon^{l*}_{\vec{k},\lambda}\epsilon^l_{\vec{k},\lambda'}\hat{a}^\dagger_{\vec{k},\lambda}\hat{a}_{\vec{k},\lambda'}$$
$$-\epsilon^l_{\vec{k},\lambda}\epsilon^{l*}_{\vec{k},\lambda'}\hat{a}_{\vec{k},\lambda}\hat{a}^\dagger_{\vec{k},\lambda'}+\mathrm{e}^{2i\omega_k t}\epsilon^{l*}_{\vec{k},\lambda}\epsilon^{l*}_{-\vec{k},\lambda'}\hat{a}^\dagger_{\vec{k},\lambda}\hat{a}^\dagger_{-\vec{k},\lambda'}\Big). \tag{45}$$

Now, combining Eqs. (38), (44) and (45) results

$$\hat{P}^n = \int d^3k\frac{\hbar\omega_k}{2c}\hat{k}^n\sum_{\lambda,\lambda'=1,2}\left(\epsilon^{m*}_{\vec{k},\lambda}\epsilon^m_{\vec{k},\lambda'}\hat{a}^\dagger_{\vec{k},\lambda}\hat{a}_{\vec{k},\lambda'}+\epsilon^m_{\vec{k},\lambda}\epsilon^{m*}_{\vec{k},\lambda'}\hat{a}_{\vec{k},\lambda}\hat{a}^\dagger_{\vec{k},\lambda'}\right). \tag{46}$$

Since $\epsilon^m_{\vec{k},\lambda}\epsilon^{m*}_{\vec{k},\lambda'}=\vec{\epsilon}_{\vec{k},\lambda}\cdot\vec{\epsilon}^{\,*}_{\vec{k},\lambda'}=\delta_{\lambda,\lambda'}$, we have

$$\hat{P}^n = \frac{1}{2}\sum_{\lambda=1,2}\int d^3k\frac{\hbar\omega_k}{c}\hat{k}^n\left(\hat{a}^\dagger_{\vec{k},\lambda}\hat{a}_{\vec{k},\lambda}+\hat{a}_{\vec{k},\lambda}\hat{a}^\dagger_{\vec{k},\lambda}\right). \tag{47}$$

Finally, we use the commutation relation $\left[\hat{a}_{\vec{k},\lambda},\hat{a}^\dagger_{\vec{k},\lambda}\right]=\delta^{(3)}(0)$,

$$\hat{P}^n = \sum_{\lambda=1,2}\int d^3k\frac{\hbar\omega_k}{c}\hat{k}^n\hat{a}^\dagger_{\vec{k},\lambda}\hat{a}_{\vec{k},\lambda}+\frac{1}{2}\delta^{(3)}(0)\sum_{\lambda=1,2}\int d^3k\frac{\hbar\omega_k}{c}\hat{k}^n. \tag{48}$$

I remind you that $\delta^{(3)}(0)=V/(2\pi)^3$, where V is the volume of the space. This is not very important because the second term in Eq. (48) vanishes; the integrand is an antisymmetric function of k^n. Therefore, recalling that $\omega_k/c=k$ and $k\hat{k}=\vec{k}$, we conclude

$$\boxed{\hat{\vec{P}} = \sum_{\lambda=1,2}\int d^3k\,\hat{a}^\dagger_{\vec{k},\lambda}\hat{a}_{\vec{k},\lambda}\hbar\vec{k}}. \tag{49}$$

This is precisely the result we should have expected. $\hbar\vec{k}$ is the momentum of a single photon with wavenumber $\vec{k}$, regardless its polarization. $\hat{a}^\dagger_{\vec{k},\lambda}\hat{a}_{\vec{k},\lambda}$ is the number density operator of photons with wavenumber $\vec{k}$ and polarization λ. We integrate over all wavenumbers, and sum over all polarizations to obtain the total momentum operator.