Final exam: AQM I, 2022-2023

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Answer all four questions. You may use only Sakurai's book, the course material from the Moodle website, and any handwritten material based on it. Consulting among each other, with others, with the internet, or with chatGPT is strictly forbidden. You can use mathematical software as appropriate. Follow instructions communicated over the course's forum. Please use e-mail to ask questions; always write to both Eytan and Yarden. You are not required to prove anything written in the course material, but you should refer to it as precisely as possible.
Your mission, should you choose to accept it: to infiltrate a secret research facility in flatland, perform several experiments and report back to us. This message will self-destruct by February 9th at midnight, by which time you must submit your answers using the website. Make sure your handwriting is legible, or you will be extradited to flatland. Good luck, Jamie/Jim.

Consider scattering in flatland (i.e., in two-dimensions):

1. Derive the spatial representation of the Lippman-Schwinger equation in 2D, i.e., write a self-consistent equation for the scattering state $\psi_{\boldsymbol{k}}^{(+)}(\boldsymbol{r})$, deriving explicitly the Green's function and the scattering amplitude.

2. Prove the optical theorem and write its explicit form. **Hint**: In flatland, the real part of the forward scattering amplitude is also contributing to the cross-section.

3. Derive the 2D "partial wave expansion" by providing an answer to all the following:

   (a) Write the incoming wave (assume it propagates along the positive x-direction) using solutions to the 2D schrodinger equation in polar coordinates (incoming and outgoing circular waves).

   (b) Add the scattered wave and derive the full asymptotic form of the scattering wavefunction.

   (c) Identify the phase shifts and use them to express the cross section in 2D.

   (d) Check your answer with the optical theorem.

4. Using the zeroth partial wave:

   (a) Find the resonance condition for the potential $\alpha\,\delta(r-r_0)$ with $\alpha>0$ (find an equation for the energy of the incoming wave that generates resonance, do not attempt to solve it).

   (b) Find the resonances numerically for $\alpha=\frac{2\hbar^2}{mr_0},\frac{3\hbar^2}{mr_0},\frac{4\hbar^2}{mr_0}$. Is there a resonance for $\alpha=\frac{\hbar^2}{mr_0}$? Explain.

In your journey, you shall find the following Bessel identities useful (Remember: Bessel functions are the radial guardians of flatland!)

$$x^2\frac{d^2Z_n(x)}{dx^2}+x\frac{dZ_n(x)}{dx}+\left(x^2-n^2\right)Z_n(x)=0 \qquad Z_n(x)=J_n(x),Y_n(x),H_n^{(1)}(x)$$

$$2\frac{dZ_n(x)}{dx}=Z_{n-1}(x)-Z_{n+1}(x)$$

$$Z_{-n}(x)=(-1)^nZ_n(x)$$

$$H_n^{(1)}(x)=J_n(x)+iY_n(x)$$

$$J_n\left(x\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi\, \mathrm{e}^{i(n\phi - x\sin\phi)}$$

$$e^{ikr\cos\phi} = \sum_{n=-\infty}^{\infty} i^n J_n\left(kr\right) e^{in\phi}$$

$$J_n\left(x\right) = \left(\frac{2}{\pi x}\right)^{1/2}\cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) + \mathcal{O}\left(x^{-1}\right)$$

$$Y_n\left(x\right) = \left(\frac{2}{\pi x}\right)^{1/2}\sin\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) + \mathcal{O}\left(x^{-1}\right)$$

$$J_{n+1}\left(x\right)Y_n\left(x\right) - J_n\left(x\right)Y_{n+1}\left(x\right) = \frac{2}{\pi x}$$

$$H_n^{(1)}(x) = \frac{2i}{\pi x^n}\int_0^{\infty}\frac{y^{n+1}J_n(y)dy}{x^2 - y^2 + i\epsilon}$$

Good luck!