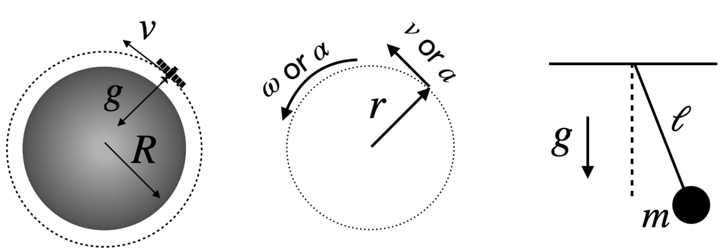
Tutorial 2

1 Dimension Analysis

Many times we can use physical quantities which characterize a system in order to guess the form of other quantities we would like to know, i.e. how they depend on the intrinsic quantities of the system. We do this by expressing each quantity in terms of basic physical dimensions: M, T, L (mass, time and length), then try to reconstruct the dimensions of the desired quantity using the others. Here are some examples:

1. **Circular Motion:**
   (a) For motion with linear tangent velocity v at radius r, express the angular velocity ω.
   (b) For motion with linear tangent acceleration a at radius r, express the angular acceleration α.
2. **Close Orbit Satellite:** A satellite in orbit with radius R close to the earth experience constant acceleration g, express the velocity v it must have in order to stay in orbit.
3. **Pendulum:** A simple pendulum, composed of a mass m attached through string with length ℓ to a static surface, undergoes oscillations. Express the frequency of the oscillations ω.



Solution:

1. The linear velocity and radius have dimensions $[v] = LT^{-1}$ and $[r] = L$, therefore, the only way to get an expression for $[\omega] = T^{-1}$ is by taking the relation v/r which means that
$$\omega = C\frac{v}{r},$$
where C is a dimensionless constant which is found to be equal 1 if we preform a more careful derivation.

2. We approximate the radius of the orbit as the radius of the earth and the acceleration to be the acceleration on the surface of the earth. Then, using the only quantities that characterize the system $[R] = L$ and $[g] = LT^{-2}$ we can easily see that the only relation to give $[v] = LT^{-1}$ is

$$v = C\sqrt{gR},$$

   here again we may find that $C = 1$ if we make the whole calculation.

3. In order to find the frequency $[\omega] = T^{-1}$ from the quantities $[\ell] = L$, $[m] = M$ and $[g] = LT^{-2}$, we see that m cannot take place in the expression since there is no other quantity to cancel its dimensions M, thus we are left with

$$\omega = C\sqrt{\frac{g}{\ell}},$$

   where, this time, C only equals to 1 for small angle approximation - that is if the oscillations are very small.

2 Gravity Acceleration

The acceleration due to gravity is, in general, not constant, but distance dependent. One can express the acceleration on a sphere (earth for example) as

$$g = g_0\left(\frac{R}{R+h}\right)^2,$$

where h is the height above sea level, $g_0 \simeq 9.8$ m/s^2 and $R \simeq 6400$ km is the radius of the earth.

1. A skydiver is jumping from height h above sea level. find his velocity as a function of height $v(y)$ (assume that he jumps from a static hot air balloon).
2. What is the velocity will his car keys, which fell from his pocket when he jumped from height $h = R$, reach sea level?
3. What would be the answers for 1-2 if you used constant acceleration $g = g_0$?

Solution:

1. Taking the expression for the acceleration can be written as

$$a = \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = \frac{dv}{dy}v.$$

   Using the expression for the acceleration g yields

$$\frac{dv}{dy}v = -g_0\left(\frac{R}{R+y}\right)^2,$$

rearranging this equation and integrating reads

$$\int v\,dv = -g_0 R^2 \int \frac{dy}{(R+y)^2}$$
$$\frac{1}{2}v^2 = g_0 \frac{R^2}{R+y} + C.$$

We can find the constant by using the initial condition $v(h) = 0$

$$C = -g_0 \frac{R^2}{R+h},$$

thus

$$v(y) = -\sqrt{2g_0 R^2 \left(\frac{1}{R+y} - \frac{1}{R+h}\right)}.$$

We can check that as y gets smaller (as the object falls) the first term increases and so does v. Also note that the units are correct.

2. Plugging in $y = 0$ and $h = R$ we find

$$v(y=0) = -\sqrt{2g_0 R^2 \left(\frac{1}{R} - \frac{1}{2R}\right)}$$
$$= -\sqrt{g_0 R},$$

using the numerical values yields a velocity of 7.92 km/s, which is ridicules since we did not account for friction with air.

3. Using constant acceleration g_0 we find

$$\int v\,dv = -g_0 \int dy$$
$$\frac{1}{2}v^2 = -g_0 y + C$$

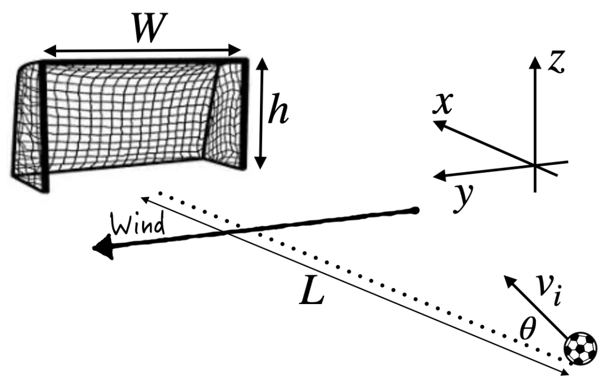
where $C = g_0 R$, thus

$$v = \sqrt{2g_0 (R-y)}.$$

We find an additional factor of $\sqrt{2}$ which accounts for the lack of lower acceleration at greater distances. Plugging in numerical values we find a velocity of 11.2 km/s.

3 Football Free Kick

A promising football player taking a penalty kick in a windy weather. The player can shoot the ball with angle $\theta = 15°$ to horizon. Given that the dimensions of a standard goal post are $h = 2.44$m height and $W = 7.32$m long, the distance from the goal line $L = 11$m and that the windy weather acts to accelerate the ball with constant acceleration of $a_w = 10$ m/s^2 in a direction parallel to the goal line (see figure):



1. What is the maximum initial velocity v_i so that player can score? (assuming that the player aims to the center of the goal post).

2. Will the player score if he shoots with initial velocity just below v_i?

3. The player decides to surprise the goal keeper by using the wind to his advantage and aims the ball with such angle that without any wind it would have hit half way between the center of the goal post and the left post (i.e. $W/4$ to the left from the center). Assuming the goal keeper won't catch the ball, will the player score?

Solution:

1. We can treat this question in 2 dimensions (x and z), we use the position equation in each direction:

$$x = v_i \cos\theta\, t,$$
$$z = v_i \sin\theta\, t - \frac{g}{2}t^2,$$

where we set the origin at the shooting post. In order to solve for v_i let us plug in t, derived from the $x = L$ equation into the $z = h$ equation

$$t = \frac{L}{v_i \cos\theta} \quad \rightarrow \quad h = L\tan\theta - \frac{g}{2}\frac{L^2}{v_i^2\cos^2\theta},$$

thus

$$v_i = \sqrt{\frac{gL^2}{2\cos^2\theta\,(L\tan\theta - h)}}.$$

We can check that for $\theta \rightarrow 90^\circ$ we find that $v_i \rightarrow \infty$ which is reasonable, we also check the numerical values to find $v_i \simeq 35.4$ m/s or 127 km/h, which is a decent velocity for a penalty kick.

2. Now we need to consider the y axis. we've found the initial velocity in 1, we could plug it into the x equation to find the time it takes the ball to reach the goal line

$$t = \frac{L}{v_i \cos\theta} \simeq 0.32\text{s}.$$

Writing down the equations for the y direction, considering the constant acceleration a_w, caused by the wind, we find

$$\Delta y = \frac{a_w}{2} t^2 \simeq 0.52\text{m},$$

that is, the ball will shift about half a meter to the left. Since $\Delta y < W/2$ we can say that the player will score.

3. We use the same calculations we did for 1-2 only change the values for the initial velocity in the x and y directions. From simply geometry we find that

$$\mathbf{v}_i = v_i \left(\cos\theta\cos\varphi, -\cos\theta\sin\varphi, \cos\theta\right),$$

where we defined $\varphi = \tan^{-1}\left(\frac{W/4}{L}\right)$. Therefore the equation for v_i we've found in 1 becomes (track the factors carefully!)

$$v_i = \sqrt{\frac{gL^2}{2\cos^2\theta\cos^2\varphi\left(L\frac{\tan\theta}{\cos\varphi} - h\right)}},$$

again we can check that for $\varphi \to 90°$ we find $v_i \to \infty$ as expected. The numerical value we find is $v_i \simeq 34.5$ m/s, which makes sense because the ball has more time to gain altitude. The corresponding time it takes the ball to reach the goal line is

$$t = \frac{L}{v_i\cos\theta\cos\varphi} \simeq 0.33\text{s},$$

which we plug into the new equation for Δy to get

$$\Delta y = v_i\cos\theta\sin\varphi\, t + \frac{a_w}{2}t^2 \simeq 2.39\text{m}.$$

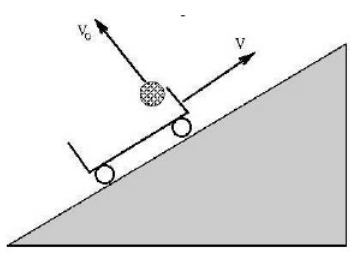
This is compared to the expected value without wind: $W/4 = 1.83$m.

4 Cart on Inclined Plain

An L length cart rising at a constant speed up an inclined plain as shown in the figure.

While the cart is moving, a ball is thrown from the top end.
The thrown velocity is V in the direction perpendicular to the plain.
Find the maximal value of v_0 for which the ball falls back into the cart.

Gravity acceleration $\boldsymbol{g}$ is given.



Solution:

Let us define a coordinate system of which the x axis is directed in the direction of the cart's velocity V and the y axis directed vertical to the plain same as v_0 direction.
The location vector of the cart is given by:

$$\boldsymbol{r_c}(t) = (Vt, 0)$$

In order to get the velocity of the ball relative to the ground we will need to add the velocity of the cart to the thrown velocity since the ball is thrown while it's already moving.
Now we need to know how to treat the gravity acceleration in our new coordinates system. Simple geometry tell us that the angle between the acceleration and the new negative y direction is the same as the angle between the inclined plain and the ground.
We denote α for the angle of inclination and define $t = 0$ for the time that the ball leaves the cart so the location vector for the ball is:

$$\boldsymbol{r_b}(t) = \left(Vt - \frac{1}{2}g\sin\alpha t^2, v_0 t - \frac{1}{2}g\cos\alpha t^2\right)$$

First we'll find the time when the ball returns to the cart by solving:

$$y_b(t) = 0 \Rightarrow v_0 t - \frac{1}{2}g\cos\alpha t^2 = 0$$

this is satisfied when $t = 0$ or $t = \frac{2v_0}{g\cos\alpha}$. We choose the later time since $t = 0$ is the start time of the motion.
In order for the ball to land in the cart the following condition must be met:

$$\left| x_c\left(\frac{2v_0}{g\cos\alpha}\right) - x_b\left(\frac{2v_0}{g\cos\alpha}\right)\right| \leq L$$

$$V\frac{2v_0}{g\cos\alpha} - \left(V\frac{2v_0}{g\cos\alpha} - \frac{1}{2}g\sin\alpha\left(\frac{2v_0}{g\cos\alpha}\right)^2\right) \leq L$$

$$\frac{2v_0^2\sin\alpha}{g\cos^2\alpha} \leq L$$

And the maximal value is when equality holds:

$$(v_0)_{MAX} = \sqrt{\frac{Lg\cos^2\alpha}{2\sin\alpha}}$$

5 Uniform Circular Motion

A particle moves in the $x - y$ plane in a circular counterclockwise motion, according to

$$\overrightarrow{r(t)} = R\left(\cos(\omega t), \sin(\omega t)\right).$$

1. Where will the particle be at $t = 0$? When will the particle return there for the first time?
2. Find $\overrightarrow{v(t)}$ and calculate its magnitude, $\left|\overrightarrow{v(t)}\right|$.
3. Calculate the scalar product $\overrightarrow{r(t)} \cdot \overrightarrow{v(t)}$.
4. Can you guess the direction of the acceleration? Find $\overrightarrow{a(t)}$ and see if you were right.
5. Repeat 2-4 using polar coordinates.

Solution:

1. Let us set $t = 0$ in $\overrightarrow{r(t)}$, $\cos(0) = 1$ and $\sin(0) = 0$, thus we get $\overrightarrow{r(t = 0)} = (R, 0)$.
In order to find when the particle returns we need to find t that meets the following conditions:

$$1. \cos(\omega t) = 1$$

$$2. \sin(\omega t) = 0$$

We deduce these conditions for t:

$$1. \omega t = 0 + 2\pi\, k$$

$$2. \omega t = 0 + \pi\, k$$

Where k is an integer. The first time we meet both conditions simultaneously after $t = 0$ is at $t = \frac{2\pi}{\omega}$.
2.

$$\overrightarrow{v(t)} = \frac{d\overrightarrow{r(t)}}{dt}$$

Use the Chain Rule to differentiate the trigonometric functions:

$$\frac{d}{dt}\left(f(\omega t)\right) = \frac{d(\omega t)}{dt}\frac{d}{d(\omega t)}\left(f(\omega t)\right)$$

We get $\overrightarrow{v(t)} = R\omega\left(-\sin(\omega t), \cos(\omega t)\right)$.

The magnitude is:

$$\sqrt{\overrightarrow{v(t)} \cdot \overrightarrow{v(t)}} = \sqrt{R^2\omega^2\left((-\sin(\omega t))(-\sin(\omega t)) + \cos(\omega t)\cos(\omega t)\right)} = R\omega\sqrt{\sin^2(\omega t) + \cos^2(\omega t)} = R\omega$$

so we get $\left|\overrightarrow{v(t)}\right| = R\omega$.

Notice! Although the velocity vector is time dependent its magnitude is independent of time, try to think why it is so.

3.

$$\overrightarrow{r(t)} \cdot \overrightarrow{v(t)} = R^2\omega\left(-\sin(\omega t)\cos(\omega t) + \sin(\omega t)\cos(\omega t)\right) = 0$$

$\overrightarrow{r(t)}$ and $\overrightarrow{v(t)}$ are orthogonal vectors as you would expect in a circular motion.

4.

$$\overrightarrow{a(t)} = \frac{d\overrightarrow{v(t)}}{dt} = R\omega^2\left(-\cos(\omega t), -\sin(\omega t)\right)$$

The direction of $\overrightarrow{a(t)}$ is opposite to the direction of $\overrightarrow{r(t)}$.

5. In polar coordinates $\rho = R$, $\phi = \omega t$ and $\overrightarrow{r(t)} = R\hat{r}$.

Calculating the time derivative of $\overrightarrow{r(t)}$:

$$\frac{d}{dt}(R\hat{r}) = \frac{dR}{dt}\hat{r} + R\frac{d\hat{r}}{dt}$$

R is not changing with time.

$\hat{\text{r}} = \cos\phi\hat{i} + \sin\phi\hat{j}$, so $\frac{d\hat{r}}{dt} = \left(-\sin\phi\hat{i} + \cos\phi\hat{j}\right)\frac{d\phi}{dt} = \omega\hat{\phi}$. and

$$\overrightarrow{v(t)} = R\omega\hat{\phi}.$$

$\hat{\phi} \cdot \hat{\phi} = 1$ and we could easily find the magnitude $\left|\overrightarrow{v(t)}\right| = R\omega$.

Since $\hat{r} \cdot \hat{\phi} = 0$ it is clear that $\overrightarrow{r(t)} \cdot \overrightarrow{v(t)} = 0$.

As before we can take a time derivative of the base vector $\hat{\phi}$: $\frac{d\hat{\phi}}{dt} = -\omega\hat{r}$.

We get the acceleration $\overrightarrow{a(t)} = -R\omega^2\hat{r}$ in an opposite direction to $\overrightarrow{r(t)}$.