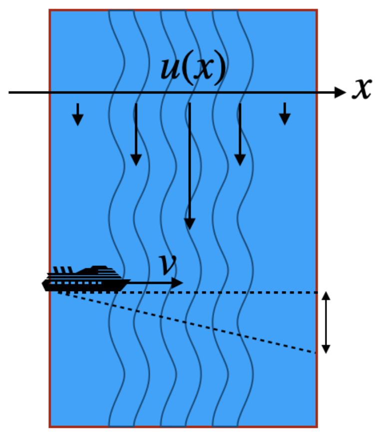
Tirgul 4

1 River Boat

A boat crosses a river perpendicular to the direction of the stream with velocity $v = 0.3$ m/s. The river is $b = 63$m wide and the velocity of the stream is described by $u(x) = u_0 \left[1 - \left(\frac{2x}{b} - 1\right)^2\right]$, where x is the distance from the bank and $u_0 = 500$ m/s. Find the deviation of the boat when it reaches the other side of the river.



Solution:

The motion is in 2 dimensions, therefore we can drop the $\hat{k}$ direction. The motion of the boat relative to the bank $\boldsymbol{v}'$ is given by the vector sum of its velocity relative to the water $\boldsymbol{v}$ and the velocity of the water relative to the bank $\boldsymbol{u}$

$$\boldsymbol{v}' = \boldsymbol{v} + \boldsymbol{u} = v\hat{\boldsymbol{\imath}} - u(x)\,\hat{\boldsymbol{\jmath}}.$$

In order to find the the position of the boat we should integrate over time

$$\begin{aligned}
\boldsymbol{r}'(t) &= \int_0^T \boldsymbol{v}' dt \\
&= \int_0^T \left(v\hat{\boldsymbol{i}} - u(x)\hat{\boldsymbol{j}} \right) dt \\
&= \int_0^b \left(v\hat{\boldsymbol{i}} - u(x)\hat{\boldsymbol{j}} \right) \frac{dt}{dx} dx \\
&= \int_0^b \left(\hat{\boldsymbol{i}} - \frac{u(x)}{v}\hat{\boldsymbol{j}} \right) dx \\
&= b\hat{\boldsymbol{i}} - \frac{u_0}{v} \int_0^b \left[1 - \left(\frac{2x}{b} - 1 \right)^2 \right] dx\, \hat{\boldsymbol{j}} \\
&= b\hat{\boldsymbol{i}} - \frac{2}{3}\frac{u_0}{v} b\hat{\boldsymbol{j}}.
\end{aligned}$$

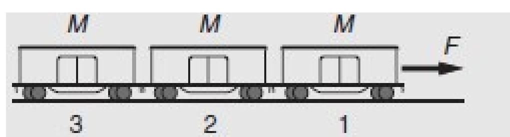
Therefore the shift along the bank is $\frac{2}{3}\frac{u_0}{v}b$.

2 Multiple Masses: a Freight Train

Three freight cars each of mass M are pulled with force F by a locomotive. Friction is negligible.

1. Find the forces on each car.
2. Consider a string of N cars, each of mass M, pulled by a force F.
   Show that the force F_n acting on the n-th car is given by:

$$F_n = \left(1 - \frac{n-1}{N} \right) F$$



Solution:

1. Because the force F is actiong on three cars with a total mass of $3M$, the acceleration of the sytem, and therefore the acceleration of each car is given by:

$$a = \frac{F}{3M}$$

   The cars are joined and are thus constrained to have the same acceleration.
   The vertical acceleration is zero, there is no vertical motion

$$Mg - N = 0 \quad \text{(true for all cars in the vertical direction)}.$$

   Let us denote F_{ij} for the force acting on the i-th car due to the adjacent j-th car.

The horizontal equation of motion for the fisrt car:

$$F - F_{12} = Ma = M\frac{F}{3M} \Rightarrow F_{12} = \frac{2}{3}F$$

For the 2nd car:

$$F_{21} - F_{23} = M\frac{F}{3M}$$

According to Newton's third law

$$F_{ij} = F_{ji}$$

we get

$$F_{23} = \frac{1}{3}F$$

And for the 3rd car:

$$F_{32} = M\frac{F}{3M} = \frac{1}{3}F = F_{23}$$

with an agreement to the third law.

2. In the case of N cars, we are looking for $F_n \to F_{n(n+1)}$ as written in this solution.
The acceleration is

$$a = \frac{F}{NM}$$

To find the force pulling the n-th car, note that the difference $F_{n(n+1)}$ and $F_{n(n-1)}$ is given by applying the second law on the n-th car for the horizontal direction:

$$F_{n(n+1)} - F_{n(n-1)} = M\frac{F}{NM} = \frac{F}{N}$$

Then, given that a force F is pulling the first car, for the n-th car we need to subtract $\frac{F}{N}$ $(n-1)$ times from F. And we get:

$$F_{n(n+1)} = F - (n-1)\frac{F}{N} = F\left(1 - \frac{n-1}{N}\right)$$

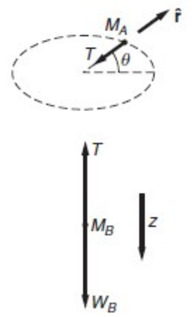
as wanted.

3 The Whirling Block

A horizontal frictionless table has a small hole in its center. Block A on the table is connected to block B hanging beneath by a string of negligible mass which passes through the hole.
Initially, B is held stationary and A rotates at constant radiusr_0 with steady angular velocity ω_0.
If B is released at t = 0, what is its acceleration immediately afterward?

Solution:

The force diagrams for A and B after the moment of release are shown in the sketches:



For A it is natural to use polar coordinates r, ϑ, while for B the linear coordinate z is sufficient, as shown in the force diagrams.
As usual, the unit vector $\hat{r}$ is radially outward.
For convenience, we have taken z to be positive in the downward direction.
The equations of motion are –

$$-T = M_A a_r \quad radial, A \tag{1}$$

$$0 = M_A a_t \quad tangential, A \tag{2}$$

$$M_B g - T = M_B a_z \quad vertical, B \tag{3}$$

Where the radial accelecration of A a_r is given by the second derivative by time of the radial coordinate of A and by the centrifual acceleration in the $-\hat{r}$ direction:

$$a_r = \ddot{r} - r\dot{\theta}^2$$

The tangential ($\hat{\phi}$ direction) acceleration of A a_t is given by:

$$a_t = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

*we can derive this expression by calculating the second derivative by time of the location vector of A $\overrightarrow{r}(t) = r(t)\hat{r(t)}$ and find the acceleration's component in the $\hat{\phi}$ direction.
And the acceleration of B is simply given by

$$a_z = \ddot{z}.$$

Because the length of the string, l, is constant, we have

$$r + z = l \tag{4}$$

Differentiating Eq. (4) twice with respect to time gives the constraint equation

$$\ddot{r} = -\ddot{z} \tag{5}$$

The negative sign means that if mass A moves outward, mass B would rise. Combining Eqs. (1), (3), and (5), we find:

$$\ddot{z} = \frac{M_B g - M_A r\dot{\theta}^2}{M_A + M_B}$$

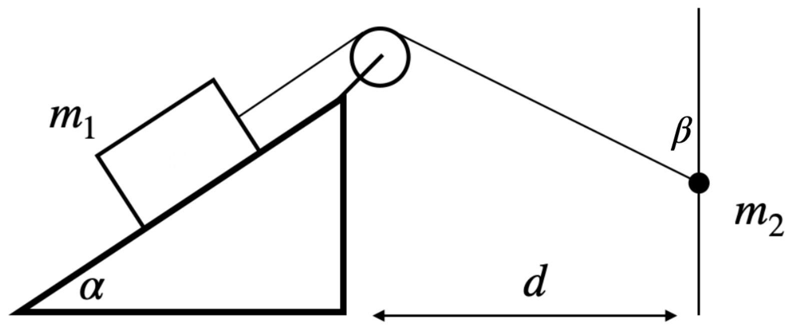
Immediately after B is released, $r = r_0 \text{and} \dot{\theta} = \omega_0$. Hence

$$\ddot{z}(0) = \frac{M_B g - M_A r_0 \omega_0^2}{M_A + M_B} \tag{6}$$

$\ddot{z}(0)$ can be positive, negative, or zero depending on the value of the numerator in Eq. (6); if ω_0 is large enough, block B will begin to rise after release.
Before release,$\ddot{r} = 0$, but immediately after, the acceleration has a finite value.
It is evident that because forces can be applied suddenly, acceleration can change abruptly—acceleration can be discontinuous in time.
In contrast, position and velocity are time integrals of acceleration and are therefore continuous in time.

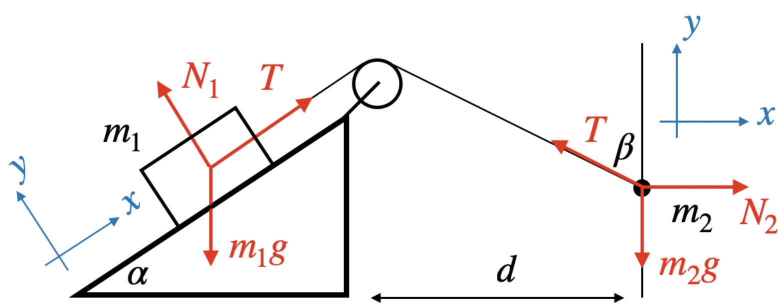
4 Bead Suspension

A body with mass m_1 lies on a slope with angle α. The body is connected to a bead with mass m_2 by a weightless rope running through and ideal pulley as shown in the figure. The bead is free to move along a vertical axis stationed at distance d from the pulley. Given that the system is at equilibrium, find the angle β between the rope and the vertical axis of the bead.



Solution:

First let us mark down all the forces that are relevant to the problem:



Next we write down the force equations for x and y axes, for each mass:

$$m_1: \quad \begin{cases} x: & T - m_1 g \sin\alpha = 0, \\ y: & N_1 - m_1 g \cos\alpha = 0, \end{cases}$$

$$m_2: \quad \begin{cases} x: & N_2 - T \sin\beta = 0, \\ y: & T \cos\beta - m_2 g = 0. \end{cases}$$

We now have 4 equations with 4 unknowns: T, N_1, N_2 and β. The first equation gives us $T = m_1 g \sin\alpha$, equations 2-3 only give us N_1, N_2 in which we have no interest at the moment. Equation 4 (along with the expression we found for T) reads

$$\beta = \cos^{-1}\left(\frac{m_2}{m_1 \sin\alpha}\right).$$

Note that the argument of the $\cos^{-1}$ is dimensionless. We also check the limits of $m_1 \gg m_2$ which corresponds to $\cos^{-1}(0) = \pi/2$, which makes sense because m_1 will pull m_2 all the way. Also, since $\cos^{-1}$ cannot get arguments greater than 1, we find maximal ration $m_2/m_1 \leq \sin\alpha$, otherwise the bead will pull the body even if it hangs in the air.