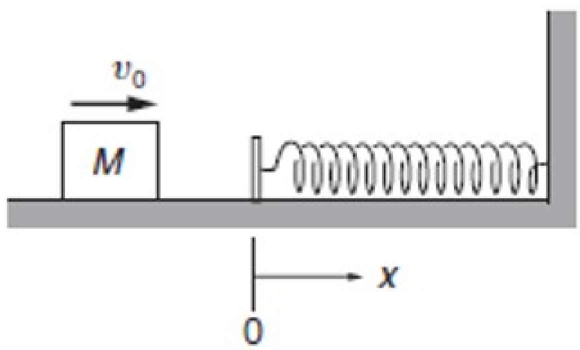
Tutorial 7

1 Block, Spring, and Friction

A block of mass M slides along a horizontal table with speed v_0.
At $t = 0$ it hits a spring with spring constant k and begins to experience a friction force, as indicated in the sketch.
The coefficient of friction is variable and is given by $\mu = bx$, where b is a constant.
Find the distance l the block travels before coming to rest.



Solution:

Initial kinetic energy:

$$K_i = \frac{1}{2}Mv_0^2$$

Final kinetic energy is vanishes because we are looking for $x = l$ where the mass velocity is zero.

$$K_f = 0$$

The force on M

$$\vec{F} = \vec{F}_{friction} + \vec{F}_{spring} = -\mu N\hat{x} - kx\hat{x}$$

where $N = Mg$ and we get

$$\vec{F} = -\left(Mgbx + kx\right)\hat{x}$$

This is a problem in 1D so the Work-Energy Theorem becomes

$$\Delta K = \int F dx$$

$$-\frac{1}{2}Mv_0^2 = -\int_0^l (Mgbx + kx)dx = -\left(\frac{Mgb + k}{2}\right)\left[x^2\right]_0^l = -\frac{Mgb + k}{2}l^2$$

$$l = \sqrt{\frac{Mv_0^2}{k + Mgb}}.$$

2 The Worm Problem

A worm of mass m and of length L rest on a horizontal table with 1 quarter of it hanging over the edge of the table.
Between the table and the worm there is a friction constant μ.
At $t = 0$ the worm is free to slip from the table.
Solve using energy considerations:

1. What will be the velocity of the worm after its entire length is detached from the table?
2. For which range of values of μ the worm will slip and not remain static?

Solution:

1. The difference in kinetic energy is
$$\Delta k = \frac{1}{2}mv_f^2$$
where v_f is the velocity of the worm when it off the table.
Three forces act on the worm: gravity (mg), friction (μN), and the normal (N).
The normal force is always perpendicular to the motion and therefore has no work.
Let us assume that the worm mass is uniform and define a longitudinal mass density $\lambda = \frac{m}{L}$.
We also define l - the length of the worm which is hanging.
The friction is working on the part which is on the table and moving horizontally
$$W_{friction} = \int_{\frac{L}{4}}^{L} \left(-\mu\lambda g(L-l)\right)\hat{x}\cdot dl\hat{x} = -\left[\mu\lambda g\left(Ll - \frac{l^2}{2}\right)\right]_{\frac{L}{4}}^{L} = -\mu\lambda g\frac{9L^2}{32} = -\frac{9}{32}\mu mgL$$
since $N = \lambda g(L-l)$.
The gravitational force is working on the part falling from the table
$$W_g = \int_{\frac{L}{4}}^{L} \left(-\lambda gl)\right)\hat{y}\cdot dl\left(-\hat{y}\right) = \lambda g\left[\frac{l^2}{2}\right]_{\frac{L}{4}}^{L} = \frac{15}{32}mgL$$
So summing both force's work and using the work - energy theorem
$$\frac{1}{2}mv_f^2 = \frac{3}{32}mgL\left(5-3\mu\right)$$
$$v_f = \sqrt{\frac{3}{16}gL\left(5-3\mu\right)}.$$
2. The worm will start to slip only if $v_f > 0$
$$5 - 3\mu > 0$$
$$\mu < \frac{5}{3}.$$

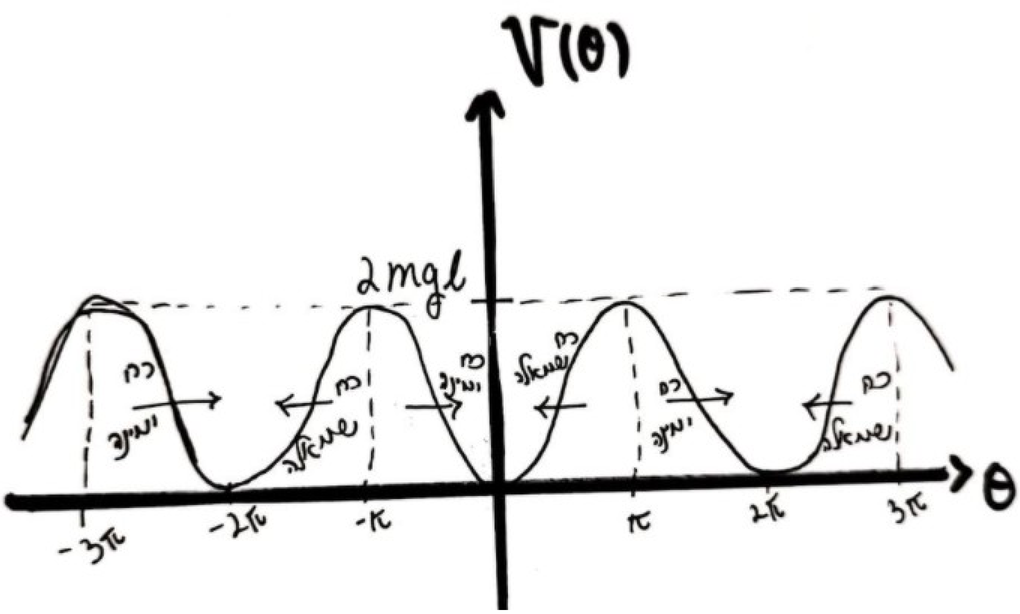
3 Pendulum's Potential

A pendulum of a thin and mass-less rod of length l ,which attached at its upper end to a static point and at its lower end to a mass m, is free to rotate around the static point.

1. What force are acting on the mass? Are they conservative?
2. What is the potential energy V of the mass, as a function of the angle of the pendulum θ?
3. Sketch $V(\theta)$.
4. Mark on the graph you have made the regions; where the force of the mass is towards $\hat{\theta}$, and where the force is towards $-\hat{\theta}$.
5. Are there equilibrium points? Are they stable?
6. What kind of trajectories the mass could have? Are they bounded (between two turning points)?
7. The kinetic energy of the mass at the bottom of the pendulum is $K(\theta = 0) = K_0$.
   If there are bounded trajectories, What is the limits on the value of K_0 for a bounded trajectory?

Solution:

1. The forces on the pendulum are the gravitational force mg - a conservative force, and the tension of the rod T.
   T is not necessarily a conservative force but since he doesn't do work on the mass it is meaningless for our goals.
2. V is the gravitational potential energy $V(\theta) = mgh(\theta) = mgl(1 - \cos\theta)$.
3. The potential can be sketched as



4. The force is related to the potential energy by

$$\overrightarrow{F} = -\frac{dV(\theta)}{ld\theta}\hat{\theta}$$

so the force is towards $\hat{\theta}$ when the slope is negative and the force is towards $-\hat{\theta}$ when the slope is positve.

5. When $\frac{dV(\theta)}{ld\theta} = 0$ the total force on the mass is zero - equlibrium.
   Let's find the equilibrium points where $-\pi \leq \theta \leq \pi$ (the physical region):

$$0 = \frac{dV(\theta)}{ld\theta} = mg\sin\theta$$

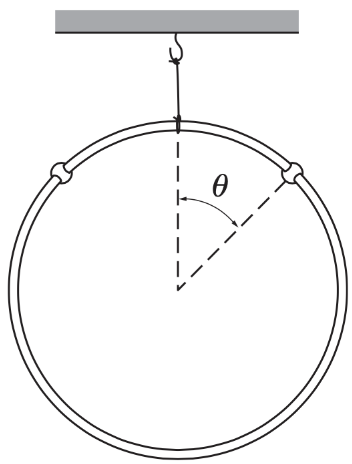
   and we get the points $\theta = 0, \pi, -\pi$.
   $\theta = 0$ is a stable equlibrium - after a little displacement, a force will bring you back to the same point. While $\theta = \pi \iff \theta = -\pi$ is unstable - after a little displacement, a force will drive the mass further away.

6. The body can perform two types of trajectories. For $E > 2mgl$ the kinetic energy is larger the the potential energy for any $\theta$and the body will preform a circular motion.
   For $E < 2mgl$ the body will oscillate around the stable point $\theta = 0$ - bounded.

7. The maximal energy for bounded trajectory is $K_0 = 2mgl$. As long as $K_0 < 2mgl$ the body will move in a closed path.

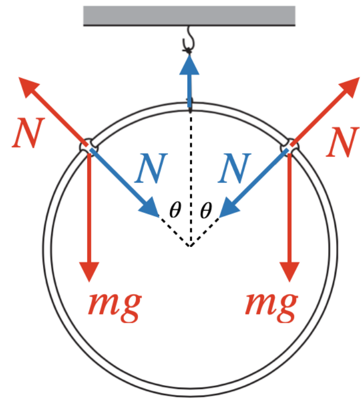
4 Beads on Hanging Ring

A ring of mass M hangs from a thread, and two beads of mass m slide on it without friction, as shown in the figure. The beads are released simultaneously from the top of the ring and slide down opposite sides. Show that the ring will start to rise if $m > 3M/2$, and find the angle at which this occurs.



Solution:

Since there is no friction, the mechanical energy is conserved. Let us mark the forces on each of the beads (red) and on the ring (blue)



The condition for the ring to remain stationary is $T \geq 0$ (since if it is not so, the string would be loose). Writing the equations of motion for the ring and the beads yields

$$T - 2N\cos\theta - Mg = 0,$$
$$N - mg\cos\theta = -m\frac{v^2}{R},$$

which comprise 4 unknown parameters: N, T, v and θ, but we are interested in the domain $T \leq 0$. Therefore we need only one additional condition, we turn to change in the kinetic energy:

$$\Delta E_k = \sum_i W_i = mgR\left(1 - \cos\theta\right)$$

$$E_k = \frac{1}{2}mv^2 = mgR\left(1 - \cos\theta\right),$$

hence

$$v^2 = 2gR\left(1 - \cos\theta\right).$$

Plugging this result into the bead's equation, we find

$$N = mg\left(3\cos\theta - 2\right),$$

using $T \leq 0$ along with the ring's equation yields

$$2N\cos\theta + Mg \leq 0 \quad \rightarrow \quad \cos^2\theta - \frac{2}{3}\cos\theta + \frac{1}{6}\frac{M}{m} \leq 0.$$

Solving for $\cos\theta$ we find

$$\cos\theta = \frac{1}{3} \pm \sqrt{\frac{1}{9} - \frac{1}{6}\frac{M}{m}},$$

which has a solution only if $\frac{1}{9} - \frac{1}{6}\frac{M}{m} \geq 0$, i.e. $m \geq \frac{3}{2}M$. In order to find θ_c we consider the solution for $\cos\theta$, we must take the $+$ sign since we are interested in the smallest θ for which the ring rises ($\cos\theta$ decrease with the increase of θ). This

$$\cos\theta = \frac{1}{3} + \sqrt{\frac{1}{9} - \frac{1}{6}\frac{M}{m}}.$$