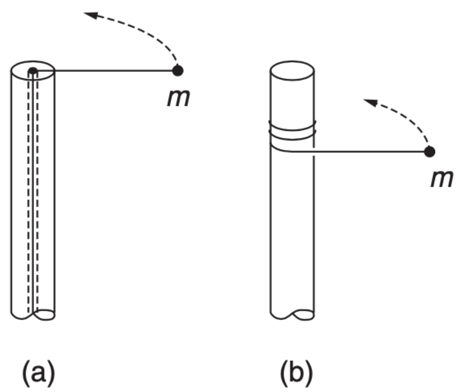
Tutorial 9

1 Mass and Post

A mass m is attached to a post of radius R by a string. Initially it is distance r from the center of the post and is moving tangentially with speed v_0.
Case (a) The string passes through a hole in the center of the post at the top. The string is gradually shortened by drawing it through the hole.
Case (b) The string wraps around the outside of the post.
What quantities are conserved in each case?
Find the final speed of the mass when it hits the post for each case.

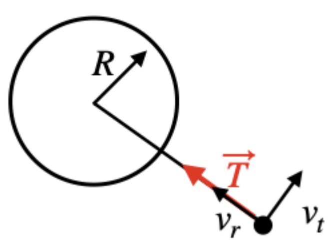


Solution:

Case (a) Since the tension force $\boldsymbol{T}$ is radial, it cannot exert a torque on m, thus the angular momentum about the center of the post is conserved. On the other hand, energy and momentum are not conserved, as the external force $\boldsymbol{T}$ pulls m closer to the post. Using the angular momentum conservation

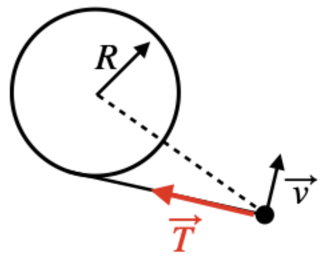
$$\boldsymbol{L}_i = \boldsymbol{r}_i \times \boldsymbol{p} = m v_0 r \hat{\boldsymbol{z}}$$
$$L_f = \boldsymbol{r}_f \times \boldsymbol{p} = m v_f R \hat{\boldsymbol{z}}$$
$$\text{thus}$$
$$v_f = \frac{r}{R} v_0,$$

where $\hat{\boldsymbol{z}}$ is the direction along the post (upwards).



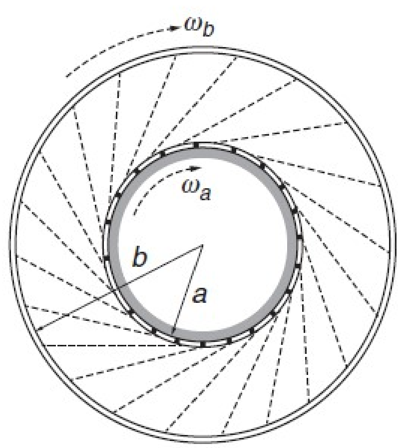
Case (b) In this case there is a component of the force $\boldsymbol{T}$ in the tangent direction, thus the angular momentum is not conserved. Momentum is not conserved either for the same reason as before. However, energy is conserved, since now $\boldsymbol{T}$ is perpendicular to the direction of motion. Therefore

$$\frac{1}{2}mv_f^2 = \frac{1}{2}mv_0^2 \quad \rightarrow \quad v_f = v_0.$$



2 Drum and Sand

A drum of mass M_A and radius a rotates freely with initial angular speed $\omega_A(0)$.
A second drum with mass M_B and radius $b > a$ is mounted on the same axis and is at rest, although it is free to rotate.
A thin layer of sand with mass M_s is distributed on the inner surface of the smaller drum.
At $t = 0$, small perforations in the inner drum are opened.
The sand starts to fly out at a constant rate $\frac{dM}{dt} = \lambda$ and sticks to the outer drum.
Find the subsequent angular velocities of the two drums ω_A and ω_B.
Ignore the transit time of the sand.



Solution:

The angular momentum of the system at time $t = 0$ is

$$L(0) = (M_A + M_s)\, a^2 \omega_A(0)$$

Note that $\omega_A(0) = \omega_A(t)$ because the sand exerts no torque on drum A as it leaves.
(To an observer on the drum, the sand appears to fly our radially.)

At time t, the angular momentum is

$$L(t) = (M_A + M_s - \lambda t)\, a^2 \omega_A(0) + (M_B + \lambda t)\, b^2 \omega_B(t)$$

Angular momentum is conserved $L(t) = L(0)$, because the system is isolated - there are no external torques.

$$\omega_B(t) = \frac{\lambda a^2 \omega_A(0) t}{(M_B + \lambda t)\, b^2}$$

At time T given by $\lambda T = M_s$, all of the sand has been transferred from drum A to drum B, and ω_B is then constant. Thus

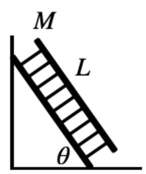
$$\omega_B(t \geq T) = \frac{M_s a^2 \omega_A(0)}{(M_B + M_s)\, b^2} < \omega_A(0)$$

It is easy to show that the angular momentum of the system for $t \geq T$ remains equal to $L(0)$.

$$\begin{aligned} L(t \geq T) &= M_A a^2 \omega_A(0) + (M_B + M_s)\, b^2 \omega_B(t \geq T) = \\ &= M_A a^2 \omega_A(0) + (M_B + M_s)\, b^2 \frac{M_s a^2 \omega_A(0)}{(M_B + M_s)\, b^2} = \\ &= (M_A + M_s)\, a^2 \omega_A(0) = L(0). \end{aligned}$$

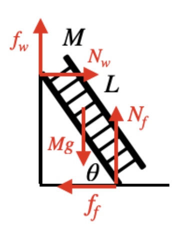
3 Ladder

A ladder with length L and mass M is placed on a floor with friction coefficients $\mu_k = \mu_s = \mu$, as it leans on a wall with similar friction coefficients. What is the minimal angle $\theta_{\min}$ for which the ladder will not slip?



Solution:

In order for the ladder to remain stationary we require that the total force and the total torque on it will vanish



$$\begin{aligned} \sum \boldsymbol{F} &= (N_w - f_f, f_w + N_f - Mg) = 0, \\ \sum \boldsymbol{\tau} &= \boldsymbol{L} \times \left(\boldsymbol{N}_w + \boldsymbol{f}_w + \frac{1}{2} M \boldsymbol{g} \right) = L \left(N_w \sin\theta + f_w \cos\theta - \frac{1}{2} Mg \cos\theta \right) = 0, \end{aligned}$$

where we chose the point of contact with the floor as the center of rotation, and clockwise rotation as positive (these choices are for convenience and do affect the physical results).
We define the moment of slipping when $f_w = \mu N_w$ and $f_f = \mu N_f$, plugging into the equations above yields

$$N_w = \mu N_f \quad N_f = Mg - \mu N_w$$

$$N_f = \frac{Mg}{1+\mu^2} \quad N_w = \frac{\mu}{1+\mu^2} Mg$$

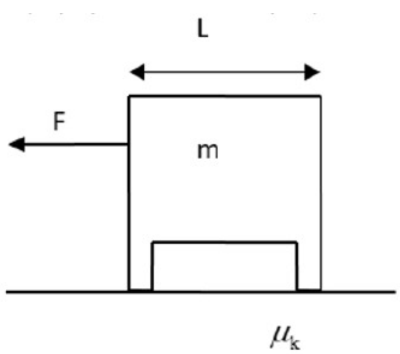
then

$$\frac{\mu}{1+\mu^2} Mg \sin\theta + \frac{\mu^2}{1+\mu^2} Mg \cos\theta - \frac{1}{2} Mg \cos\theta = 0 \quad \rightarrow \quad \tan\theta_{\min} = \frac{1-\mu^2}{2\mu}.$$

4 Dinner Table

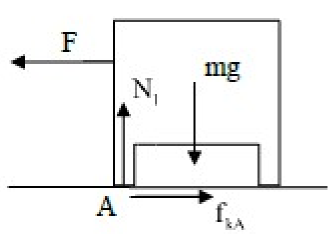
A guy is interested in moving its dinner table of length L and of mass m.
In order to do so, he pulling the table using horizontal force $\boldsymbol{F}$.
Between the table and the floor there is a friction coefficient μ_k.

1. At what maximum height h_{max} from the floor it is possible to pull the table without it being overturned?
2. For some $h < h_{max}$, find the normal forces on each leg of the table?



Solution:

1. If the table will turn over its rotating axis will be on its left leg. Which means that his right leg will need to left the floor and therefore no normal force will act on it.
   For the moment before the overturn $N_2 = 0$:



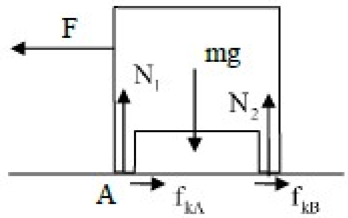
   The torque equation on point A

$$\Sigma\tau_A = Fh - \frac{mgL}{2} = 0$$

   And we get

$$h_{max} = \frac{mgL}{2F}.$$

2. Now $N_2 \neq 0$ so



The force equations on the center of mass:

$$\Sigma F_y = N_1 + N_2 - mg = 0$$

and the torque equation

$$\Sigma \tau_A = Fh - \frac{mgL}{2} + N_2 L = 0$$

solving using algebra

$$N_2 = \frac{1}{2}mg - \frac{h}{L}F,$$

$$N_1 = \frac{1}{2}mg + \frac{h}{L}F.$$