Tutorial 10

1 Moment of Inertia

1. A rod with length L is located along the x axis so that its left edge is at the origin (its right edge is at $x = L$). Given that the rod's mass density is $\lambda(x) = \lambda_0 \frac{x^2}{L^2}$.
   (a) Calculate the moment of inertia of the rod relative to the origin (as it rotates in the $x - y$ plane).
   (b) Calculate the moment of inertia of the rod relative to the point $(L, 0)$ (as it rotates in the $x - y$ plane).
2. Calculate the moment of inertia for a thin sphere of mass M and of radius R around an axis passing trough the sphere's center.

Solution:

1. Following the definition of moment of inertia for rigid bodies $I = \int r^2 dm$, we solve
   (a) Rotation around the origin yields the following moment of inertia

$$I = \int r^2 dm = \int x^2 dm = \frac{\lambda_0}{L^2} \int_0^L x^4 dx = \frac{\lambda_0 L^3}{5}.$$

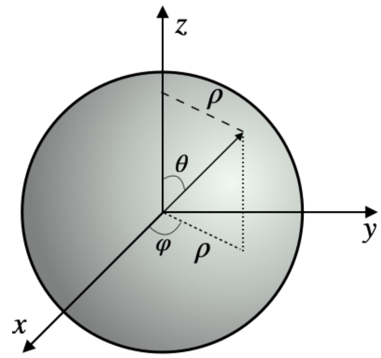
   (b) Rotation around $(L, 0)$ yields the following moment of inertia

$$\begin{aligned} I = \int r^2 dm &= \int (L - x)^2 \, dm \\ &= \frac{\lambda_0}{L^2} \int_0^L (L - x)^2 x^2 dx \\ &= \frac{\lambda_0}{L^2} \int_0^L \left(L^2 x^2 - 2Lx^3 + x^4\right) dx \\ &= \lambda_0 L^3 \left(\frac{1}{3} - 2\frac{1}{4} + \frac{1}{5}\right) \\ &= \frac{\lambda_0 L^3}{30} \end{aligned}$$

2. Due to spherical symmetry, it does not matter around which axis that goes trough the center we will calculate. Let's choose the z axis.

$$I_{zz} = \int \left(x^2 + y^2\right) dm = \int \rho^2 \sigma dS,$$

where ρ - the distance from the z axis, σ - is the mass density per unit area and dV - volume element.



In a uniform spherical shell:
$\rho = R\sin\theta$, $\sigma = \frac{M}{4\pi R^2}$, and $dS = R^2 \sin\theta d\theta d\varphi$, and the limits on the angles are

$$\theta : 0 \to \pi, \quad \varphi : 0 \to 2\pi,$$

thus

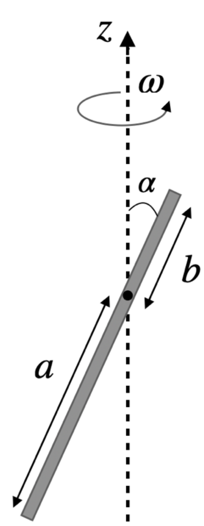
$$I_{zz} = \int_0^{2\pi}\int_0^{\pi} (R\sin\theta)^2 \frac{M}{4\pi \cancel{R^2}} \cancel{R^2} \sin\theta d\theta d\varphi =$$

$$\{\text{Integrating } \varphi \Rightarrow 2\pi\} = \frac{R^2}{2}\int_0^{\pi} \left[1-\cos^2\theta\right] \sin\theta d\theta = \{u \to \cos\theta, du \to -\sin\theta d\theta, u : 1 \to -1\} =$$

$$= \frac{R^2}{2}\int_{-1}^{1} \left[1-u^2\right] du = \frac{MR^2}{2}\left[u - \frac{u^3}{3}\right]_{-1}^{1} = \frac{2}{3}MR^2.$$

2 Rotating Rod

A thin rod of mass M and length $a+b$, $b < a$, rotate around z axis with angular frequency ω, around point O which is in distance b from the top.
What should be the angle between the rod and z axis?



Solution 1:

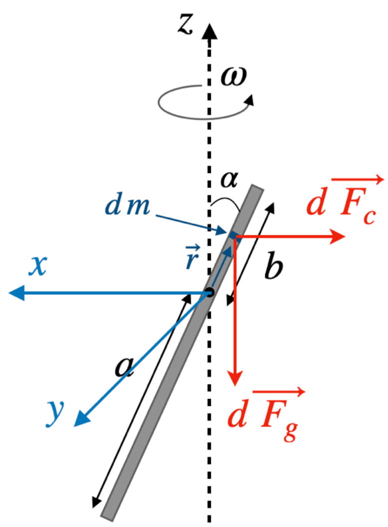
In order for the rod to rotate with a steady angle α with the z axis, its total torque around the y axis, from the fixed point outwards from the screen, should be zero.
In the frame rotating with the rod (simplify the calculations) the forces acting on a rod element are

- The centrifugal force

$$d\boldsymbol{F_c} = -dm\omega^2 l \sin\alpha \hat{\boldsymbol{x}}$$

- The gravitational force

$$d\boldsymbol{F_g} = -dmg\hat{\boldsymbol{z}}$$



The total force on element of the rod

$$\Sigma d\boldsymbol{F} = \left(-\omega^2 l \sin\alpha, 0, -g\right) dm$$

The vector from the fixed point of the rod to the rod element

$$\boldsymbol{r} = l\left(-\sin\alpha, 0, \cos\alpha\right)$$

Then the element torque around the fixed point:

$$d\boldsymbol{\tau} = l \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ -\sin\alpha & 0 & \cos\alpha \\ -\omega^2 l \sin\alpha & 0 & -g \end{vmatrix} dm = -\hat{\boldsymbol{y}} l \left(g\sin\alpha + \omega^2 l \sin\alpha\cos\alpha\right) dm$$

where $dm = \lambda dl = \frac{M}{a+b} dl$ and $l : -a \to b$.
We integrate to find $\boldsymbol{\tau}$ and equate to zero,

$$0 = \frac{M\sin\alpha}{a+b} \int_{-a}^{b} \left(gl + \omega^2 \cos\alpha l^2\right) dl = \frac{M\sin\alpha}{a+b} \left(\frac{g}{2}\left[l^2\right]_{-a}^{b} + \frac{\omega^2\cos\alpha}{3}\left[l^3\right]_{-a}^{b}\right)$$

$$0 = \frac{M\sin\alpha}{a+b}\left(\frac{g}{2}\left(b^2 - a^2\right) + \frac{\omega^2\cos\alpha}{3}\left(b^3 + a^3\right)\right).$$

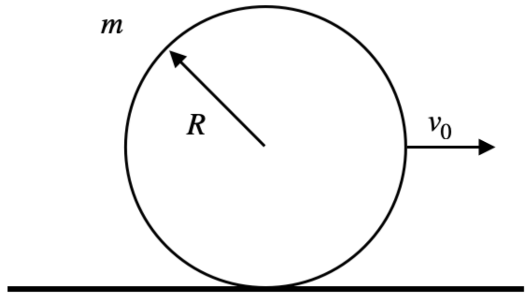
We can discard the solution $\sin\alpha = 0$ which corresponds to $\alpha = 0, \pi$. (the rod is aligned with the z axis), leaving us with

$$\frac{g}{2}\left(a^2 - b^2\right) = \frac{\omega^2\cos\alpha}{3}\left(a^3 + b^3\right),$$

$$\cos\alpha = \frac{3g}{2\omega^2}\frac{a^2 - b^2}{a^3 + b^3}.$$

3 Slip and Roll

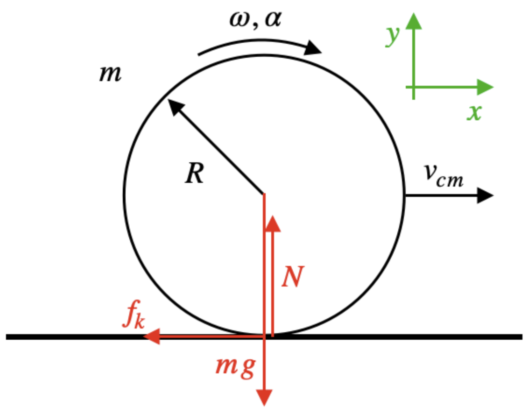
A ball with mass m and radius R is moving on a flat surface, with friction coefficients $\mu_k = \mu_s = \mu$, with initial velocity v_0 (not rolling).



1. What is the distance the ball passes till it rolls without slipping?
2. What is the work done by the friction force, to what distance the friction force does work on the ball?

Solution:

Once the ball begins its motion it experiences a kinetic friction force with exerts torque on the ball, causing it to roll.



1. Let us write the force equations in both axes

$$x: \quad \sum F_x = -f_k = m\left(a_{cm}\right)_x$$
$$y: \quad \sum F_y = N - mg = 0.$$

While the torque equation yields

$$\sum \tau = Rf_k = I\alpha,$$

where $f_k = \mu N = \mu mg$ and the moment of inertia of a ball is given by $I = 2mR^2/5$. So that

$$a_{cm} = -\mu g \quad \rightarrow \quad v_{cm}(t) = v_0 - \mu g t,$$

and

$$\alpha = \frac{Rf_k}{I} = \frac{5}{2}\frac{\mu g}{R} \quad \rightarrow \quad \omega(t) = \frac{5}{2}\frac{\mu g}{R}t.$$

Roll without slipping occurs when the bottom point of the ball is stationary relative to the surface, this happens only if this point moves with the same linear velocity as that of the center of mass of the ball, i.e. $\omega R = v_{cm}$. Therefore the time it takes the ball to stop slipping t' is

$$\frac{5}{2}\frac{\mu g}{R}t'R = v_0 - \mu g t' \quad \rightarrow \quad t' = \frac{2}{7}\frac{v_0}{\mu g}.$$

The distance the ball covers by that time is

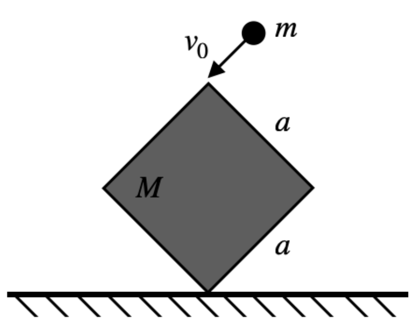
$$x_{cm}(t') = v_0 t' - \frac{\mu g}{2}t'^2 = \frac{2}{7}\frac{v_0^2}{\mu g} - \frac{2}{49}\frac{v_0^2}{\mu g} = \frac{12}{49}\frac{v_0^2}{\mu g}.$$

2. The work done by the friction force is $W_f = \int \boldsymbol{f}_k \cdot d\boldsymbol{l}$, where $d\boldsymbol{l}$ is the differential distance over which the force acts. If the ball was only slipping, then we'd expect $d\boldsymbol{l} = d\boldsymbol{x}$, but since the ball also rolls, the distance is less than the linear translation of the center of mass (think about the scenario for which the ball rolls with a tangent velocity equals to v_{cm}, i.e. no slipping, for which the friction force does no work at all since there is no relative motion between the bottom of the ball and the surface). Instead, the distance $d\boldsymbol{l}$ is equal to the subtraction $(dx - ds)\hat{\boldsymbol{x}}$, where ds is the distance covered by the rotation of the ball. Therefore

$$\begin{aligned} W_f = \int \boldsymbol{f}_k \cdot d\boldsymbol{l} &= -\int \mu m g\,(dx - ds) \\ &= -\int \mu m g\,(v_0 - \mu g t - \omega(t) R)\,dt \\ &= -\int_0^{t'} \mu m g\left(v_0 - \mu g t - \frac{5}{2}\mu g t\right)dt \\ &= -\mu m g\left(v_0 t' - \frac{7}{4}\mu g t'^2\right) \\ &= -\frac{m v_0^2}{7}. \end{aligned}$$

4 Square Disk

A square, flat, uniform disk, with side length a and mass M is positioned on a surface with very high friction at an unstable equilibrium (see figure). At time $t = 0$ a mass m hits the top corner of the disk, with velocity v_0 parallel to the side of the disk (see figure, again). After the collision the mass sticks to the corner. It is also given that the moment of inertia of such disk around its center of mass is $I_{cm} = \frac{1}{6}Ma^2$.
What would be the velocity of the top corner the moment after the collision?



Solution:

We will solve the rotation relative to the bottom corner of the disk (the one in contact with the surface), since it will eliminate the need of accounting for the torque due to the normal and friction forces. First let us find the moment of inertia of the disk relative to this point, using Steiner's theorem $I = I_{cm} + Mr^2$, we take

$$I = I_{cm} + M\left(\frac{a}{\sqrt{2}}\right)^2 = \frac{2}{3}Ma^2.$$

Next we use the conservation of angular momentum during the collision, before the collision the total momentum is only due to m

$$\boldsymbol{p} = -\frac{mv_0}{\sqrt{2}}\left(\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}\right),$$

thus the total angular momentum is

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = \boldsymbol{r} = \sqrt{2}a\hat{\boldsymbol{y}} \times -\frac{mv_0}{\sqrt{2}}\left(\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}\right) = mv_0a\hat{\boldsymbol{z}}.$$

Since it is conserved we can write

$$L_f = mv_f\sqrt{2}a + I\omega = mv_0a \quad \rightarrow \quad mv_0a = mv_f\sqrt{2}a + \frac{2}{3}Ma^2\frac{v_f}{\sqrt{2}a} = v_f\left(\sqrt{2}ma + \frac{\sqrt{2}}{3}Ma\right)$$

hence

$$\boldsymbol{v}_f = \frac{mv_0a}{\sqrt{2}ma + \frac{\sqrt{2}}{3}Ma}\left(-\hat{\boldsymbol{x}}\right) = -\frac{\sqrt{2}v_0}{2 + \frac{2}{3}\frac{M}{m}}\hat{\boldsymbol{x}}.$$