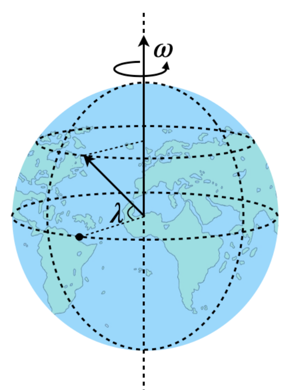
Tutorial 11

1 Rock on Earth

A rock is placed on the ground of the earth, which is rotating around its axis with angular velocity $\omega = 7.292 \times 10^{-5} \frac{\text{rad}}{\text{sec}}$ and has radius of $r = 6.35 \times 10^6$m. Earth's gravitational acceleration is $\boldsymbol{g}_0$ ($g_0 = 9.8 \frac{m}{s^2}$). The stone is located at latitude λ, at rest relative to the earth.



1. Someone lifts the rock and then drops it. What is the effective gravitational acceleration $\boldsymbol{g}$ that the rock experiences?
2. What are the radial and tangent (along the longitudinal line) components of the centrifugal acceleration? What is the deviation of the angle from the radial direction due to the effective gravitational acceleration?
3. Now, the rock falls freely. What is the magnitude and direction of the Coriolis acceleration that the rock experiences? In which direction would it deviate if it is at the northern/southern hemisphere?
4. Now, the rock is thrown horizontally (tangent to the earth's surface) with constant velocity $\boldsymbol{v}$. What is the Coriolis acceleration in this case?

Solution:

1. Writing down the expression for acceleration in a rotating frame:

$$\boldsymbol{a}' = \boldsymbol{a} + 2\boldsymbol{\Omega} \times \boldsymbol{v} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) \, ,$$

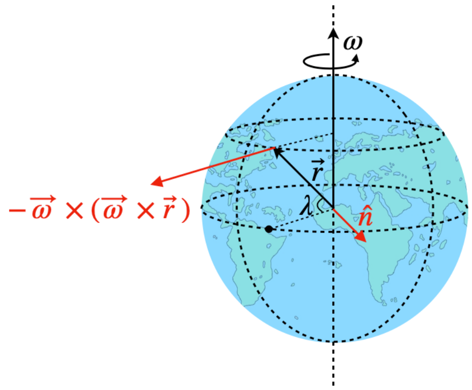
where $\boldsymbol{a}'$ is the acceleration relative to stationary frame (not rotating with the earth) and $\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{v}$ are relative to the earth's frame. In our case, $\boldsymbol{a} = \boldsymbol{g}_0$ and $\boldsymbol{\Omega} = \boldsymbol{\omega}$, hence

$$\boldsymbol{g}_0 = \boldsymbol{a} + 2\boldsymbol{\omega} \times \boldsymbol{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) \quad \rightarrow \quad \boldsymbol{a} = \boldsymbol{g}_0 - 2\boldsymbol{\omega} \times \boldsymbol{v} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) \, .$$

Since the rock's initial velocity relative to the earth is 0, the second term vanishes initially, thus

$$\boldsymbol{a} = \boldsymbol{g}_0 - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) \, ,$$

i.e. the effective acceleration will be smaller than g_0, since $\boldsymbol{g}_0$ points into the earth while the second term points radially.



Let us evaluate the second term,

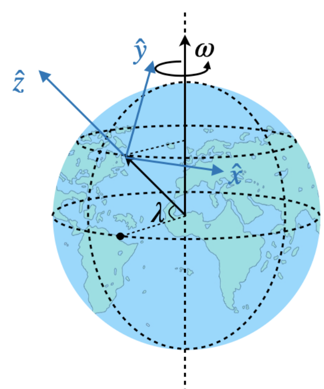
$$\boldsymbol{\omega}\times\boldsymbol{r} = \omega r\sin\left(\frac{\pi}{2}-\lambda\right)\hat{\boldsymbol{n}} = \omega r\cos\lambda\hat{\boldsymbol{n}},$$

where $\hat{\boldsymbol{n}}$ points perpendicular to the $\boldsymbol{\omega}-\boldsymbol{r}$ plane. thus

$$\boldsymbol{a}_c \equiv -\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}) = \omega^2 r\cos\lambda\hat{\boldsymbol{\rho}},$$

note that $a_c \sim \omega^2 r \approx 10^{-2}\frac{m}{s^2} \ll g_0$, also as $\lambda\to\pm\pi/2$ (at the poles) the centrifugal acceleration vanishes.

2. Let us define cartesian coordinate system with an origin at the rock, x points to the east, y points to the north and z to the sky (radial direction),



Note that λ is now the angle between $\boldsymbol{a}_c$ and $\hat{\boldsymbol{z}}$, so that

$$\begin{aligned}(a_c)_z &= a_c\cos\lambda = \omega^2 r\cos^2\lambda,\\ (a_c)_y &= -a_c\cos\left(\frac{\pi}{2}+\lambda\right) = -\omega^2 r\cos\lambda\sin\lambda,\end{aligned}$$

so that $(a_c)_y$ points south when the stone is at the northern hemisphere ($\lambda > 0$) and vice versa. Evaluating the components of the effective gravitational acceleration $\boldsymbol{g}$ yields

$$\begin{aligned}\boldsymbol{g}_z &= -g_0 + \omega^2 r\cos^2\lambda,\\ \boldsymbol{g}_y &= -\omega^2 r\cos\lambda\sin\lambda,\end{aligned}$$

thus the deviation angle from the radial direction is

$$\tan\alpha = \frac{|g_y|}{|g_z|} = \left|\frac{\omega^2 r\cos\lambda\sin\lambda}{g_0-\omega^2 r\cos^2\lambda}\right|,$$

where $\alpha\approx 0$ due to the relation $g_0 \gg \omega^2 r$.

3. In a free fall scenario, $\boldsymbol{v} \neq 0$ and points into the earth (as we found, we may neglect the centrifugal contribution to the direction of the effective acceleration). In the (x, y, z) system we defined before, the Coriolis acceleration reads

$$-2\boldsymbol{\omega} \times \boldsymbol{v} = -2 \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ 0 & \omega \cos\lambda & \omega \sin\lambda \\ v_x & v_y & v_z \end{vmatrix} = 2\omega \begin{pmatrix} v_y \sin\lambda - v_z \cos\lambda \\ -v_x \sin\lambda \\ v_x \cos\lambda \end{pmatrix} = 2\omega \begin{pmatrix} v \cos\lambda \\ 0 \\ 0 \end{pmatrix} = 2\omega v \cos\lambda \hat{\boldsymbol{x}},$$

   hence the Coriolis acceleration points towards the east if the rock falls ($v_z = -v$). Therefore, considering the centrifugal acceleration as well we find that the rock will accelerate:
   Northern hemisphere - towards south-east.
   Southern hemisphere - towards north-east.

4. We use the same result in (3) with $\boldsymbol{v} = (v_x, v_y, 0)$, to find

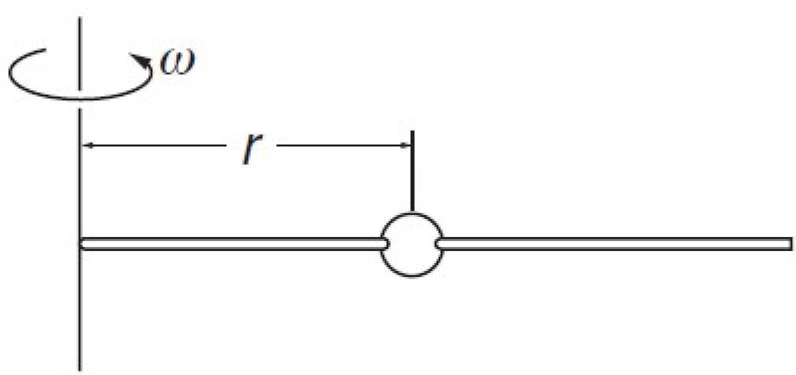
$$-2\boldsymbol{\omega} \times \boldsymbol{v} = 2\omega \begin{pmatrix} v_y \sin\lambda \\ -v_x \sin\lambda \\ v_x \cos\lambda \end{pmatrix},$$

   thus the acceleration has both horizontal and vertical components (tangent and radial). Note that the horizontal acceleration points south-east at the northern hemisphere and north-west at the southern hemisphere (for $v_x, v_y > 0$, i.e. motion towards north-east), also it is notable that the horizontal (tangent) acceleration vanishes at the equator ($\lambda = 0$) and the vertical (radial) component $\sim \omega v_x \approx 10^{-4} \left[s^{-1}\right] v_x$ is negligible for any velocity $v_x \lesssim 10^4 \frac{m}{s}$ (corresponds to 36×10^3 km/h which is ridiculous).

2 Bead in Rotating Frame

A bead slides on a horizontal rod which rotates at an constant angular velocity ω with no friction between the bead and the rod.
Neglect gravity.

1. Write down the equations of forces for the bead.
2. Calculate the normal force acting on the bead from the rod. You can leave 2 degrees of freedom.



Solution:

Working with a cylindrical coordinate system rotating with the bead.
The angular velocity vector

$$\boldsymbol{\omega} = \omega \hat{\boldsymbol{z}}.$$

In the frame of the rotating rod, ignoring gravity, we find

$$m\ddot{\boldsymbol{r}} = \boldsymbol{N} + 2m\dot{\boldsymbol{r}} \times \boldsymbol{\omega} + m\left(\boldsymbol{\omega} \times \boldsymbol{r}\right) \times \boldsymbol{\omega}$$

where $\boldsymbol{r} = r\hat{\boldsymbol{r}}$ and $\dot{\boldsymbol{r}} = \dot{r}\hat{\boldsymbol{r}}$. Calculating explicitly yields the forces:
Coriolis

$$-2m\dot{r}\omega\hat{\boldsymbol{\varphi}}$$

Centrifugal

$$m\omega^2 r\hat{\boldsymbol{r}}$$

There is another force since the rod and the bead touching - Normal force.
In a general form

$$\boldsymbol{N} = N_r\hat{\boldsymbol{r}} + N_\varphi\hat{\boldsymbol{\varphi}} + N_z\hat{\boldsymbol{z}},$$

but since we ignore gravity on the $\hat{\boldsymbol{z}}$ direction, which is naturally countered by N_z, we will find no z component, also we know that physically there is no radial component to the normal since the bead may slide freely along the rod. Thus

$$\boldsymbol{N} = N\hat{\boldsymbol{\varphi}}.$$

1. The forces equations (Using Newton's second law):

$$\Sigma F_r = m\omega^2 r = m\ddot{r}$$

$$\Sigma F_\varphi = N - 2m\omega\dot{r} = 0$$

2. Using these two equation, we find

$$N = 2m\omega\dot{r},$$

thus the normal N is proportional to the radial velocity of the bead. We can find $\dot{r}$ by solving the 2nd order differential equation for r

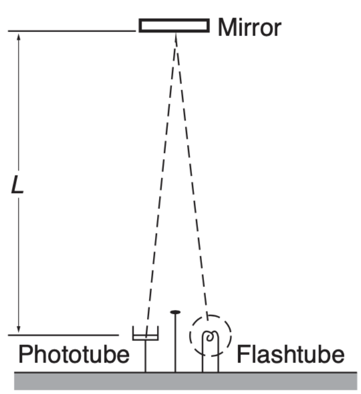
$$\ddot{r} - \omega^2 r = 0,$$

which yields 2 degrees of freedom (initial conditions for r and $\dot{r}$).

3 Time Dilation

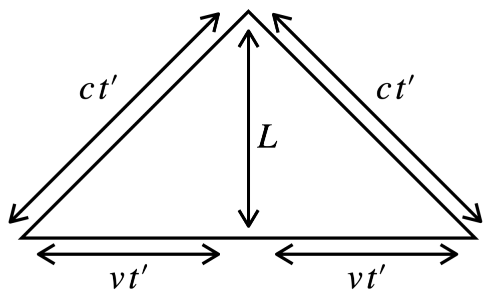
The clock in the sketch can provide an intuitive explanation of the time dilation formula. The clock consists of a flashtube, mirror, and phototube. The flashtube emits a pulse of light that travels distance L to the mirror and is reflected back to the phototube. Every time a pulse hits the phototube it triggers the flashtube. Neglecting time delay in the triggering circuits, the period of the clock is $\tau_0 = 2L/c$. Now examine the clock in a coordinate system moving to the left with uniform velocity v. In this system the clock appears to move to the right with velocity v. Find the period of the clock in the moving system by direct calculation, using only the assumptions that c is a universal constant, and that distance perpendicular to the line of motion is unaffected by the motion. The result should be identical to that given by the Lorentz transformation:

$$\tau = \tau_0/\sqrt{1 - v^2/c^2}.$$



Solution:

In the moving frame S' the motion of the light is described by the triangle:



Therefore

$$(ct')^2 = L^2 + (vt')^2 \quad \rightarrow \quad (t')^2 = \frac{L^2}{c^2 + v^2}.$$

The total time it takes for the light to complete the period is

$$\tau = 2t' = 2\frac{L}{c}/\sqrt{1 + v^2/c^2} = \tau_0/\sqrt{1 + v^2/c^2}.$$

4 Relativistic Sesame

Ernie and Bert moving in a velocity of $0.6c$ relative to each other.
At $t = t_0 = 0$ they both meets in the same place.
After 5 years according to its clock, Ernie sends light signal to Bert.

1. How long it will take for the signal to reach Bert according Ernie's clock?
2. How long according to Bert's clock it takes to (a) Ernie send the signal? (b) The signal to arrive?
3. According to each of them what will be the distance between the two in the moment when the signal reaches Bert?

Solution:

Denote A - The event when they both meets, B - Event when Ernie sends a light signal and C - The signal reaches Bert.
We note that $\beta = 0.6$ and the Lorentz factor is $\gamma = \frac{1}{\sqrt{1-0.6^2}} = \frac{5}{4}$.
The measurements of Ernie are measured in system S and the measurements of Bert are measured in system S'.
For event A

$$x_A = 0 \;\; t_A = 0 \;\text{ and also }\; x'_A = 0 \;\; t'_A = 0.$$

For event B which is happened in Ernie location

$$x_B = 0 \;\; t_B = 5\,years$$

Using Lorentz transformation

$$x'_B = \gamma\left(x_B - \beta c t_B\right) = \frac{5}{4}\left(0 - 0.6 \cdot 5\,light\,years\right) = -3.75\,light\,years$$

$$t'_B = \frac{\gamma}{c}\left(ct_B - \beta x_B\right) = \frac{5}{4}\left(5\,light\,years - 0\right) = \underline{6.25\,years} \text{ (answer to 2(a))}.$$

For event C which is happend in Bert loaction - the signal (which in each system advances at the speed of light) passed a distance of $3.75\, light\, years$ from where it was in event B.
Therefore

$$t'_C = 6.25 + 3.75 = \underline{10\, years}(\text{answer to 2(b)}) \;\; x'_C = 0.$$

Using Lorentz tranformation

$$t_C = \frac{\gamma}{c}\left(ct'_C + \beta x'_C\right) = \frac{5}{4}\left(10 + 0\right) = \underline{12.5\, years}(\text{answer to 1}).$$

$$x_C = \gamma\left(x'_C + \beta ct'_C\right) = \frac{5}{4} 0.6c \cdot 10\, years = 7.5\, light\, years.$$

According to Ernie the distance between the two is $7.5\, light\, years$.
If Ernie's loaction when the signals arrives is $x = 0$ and its clock shows $t = 12.5\, years$, so their distance according to Bert will be given by

$$\left|\gamma\left(0 - \beta ct\right)\right| = \left|\frac{5}{4}\left(-0.6 \cdot 12.5\, light\, years\right)\right| = 9.375\, light\, years$$