Statistical Mechanics - Class Exercise 5

December 5, 2022

The Boltzmann distribution function

If we want to calculate the flux of particles with velocity v

$$J_v = \iint_{\theta < \frac{\pi}{2}} \frac{d\Omega}{4\pi} \frac{N}{V} v \cos(\theta) = \frac{1}{2} \int_{\theta}^{\frac{\pi}{2}} d\theta \sin(\theta) \frac{N}{V} v \cos(\theta) = \frac{1}{4} \left(\frac{N}{V} \right) v$$

The number of particles

$$\begin{aligned} N = & \iint d\boldsymbol{r} \frac{d\boldsymbol{p}}{(2\pi)^3} f(\epsilon_{\boldsymbol{p}} - \mu) \\ = & V \left(\frac{m}{2\pi} \right)^3 \int d\boldsymbol{v} f \left(\frac{mv^2}{2} - \mu \right) \\ = & V \left(\frac{m}{2\pi} \right)^3 \int 4\pi v^2 dv f \left(\frac{mv^2}{2} - \mu \right) = \int F(v) \, dv \end{aligned}$$

$$F(v) = V \left(\frac{m}{2\pi} \right)^3 4\pi v^2 f \left(\frac{mv^2}{2} - \mu \right)$$

So the total flux is

$$J = \int_0^\infty \frac{1}{4} \left(\frac{F(v) \, dv}{V} \right) v$$

Exercise 6030 - Thermionic emission of electrons from a metal

A spherical piece of metal (“cathode”), that has radius R and temperature T, is placed inside a vacuum tube. A second metallic piece (”anode”) is used to collect the electrons that are emitted from the cathode. The effective temperature of the anode is zero. The cathode has a work function W, while the anode has work function W'. The depth of the potential that holds the electrons inside the cathode, aka the potential floor, is V_0.

1. Write an integral expression for the saturation current I_s that would be measured if the bias voltage is very large.

(a) Show that V_0 does not appear in the final result: the outcome of the calculation is the same for sections that are close to the surface or deep in the metal.

(b) Calculate the integral using the Boltzmann approximation. Specify the range of temperatures for which the approximation is valid.

2. Using the result of the previous item write an estimate for the current if a reverse (stopping) voltage V_{battery} is applied. Explain whether W or W' is relevant.

(a) Explain the relation to the analysis of the stopping voltage in the photoelectric effect.

3. Assume that the cathode is detached and left alone in free space. Calculate the charge $Q(t)$ of the cathode as a function of time assuming that $Q(0) = 0$.

(a) Explain the limitations of the result that you have obtained.

Answer

1. The energy of electron on the surface of the cathode is

$$E = \frac{mv^2}{2} - eV_0$$

The velocity $v^2 = v_\parallel^2 + v_\perp^2$, Where $v_\parallel$ and $v_\perp$ are the electron velocities in the direction parallel and perpendicular to the surface respectively.
The minimum energy for electron to escape from the cathode is when $v_\parallel = 0, v_\perp = \sqrt{\frac{2eV_0}{m}}$. For a large (repelling) bias voltage, all escaping electrons reach the anode and contribute to the current. We want to calculate he total flux, but because the different in the minimum of the velocity in the different directions let's define $F(\boldsymbol{v})$such that

$$\int F(\boldsymbol{v})\, d\boldsymbol{v} = \int F(v)\, dv$$

So $F(\boldsymbol{v}) = 2 \times V \times \left(\frac{m}{2\pi}\right)^3 f\left(\frac{mv^2}{2} - \mu\right)$,

$$J = \int \left(\frac{F(\boldsymbol{v})\, d\boldsymbol{v}}{V}\right) v_\perp$$

$$J = 2\left(\frac{m}{2\pi}\right)^3 \iint d^2 v_\parallel \int_{\sqrt{\frac{2eV_0}{m}}}^{\infty} dv_\perp f\left(\frac{mv^2}{2} - \mu\right) v_\perp$$

Where $\mu = eV_0 - W$

$$J = \left(\frac{m}{2\pi}\right)^3 \iint_{-\infty}^{\infty} d^2 v_\parallel \int_{\frac{2eV_0}{m}}^{\infty} dv_\perp^2 f\left(\frac{mv_\parallel^2}{2} + \frac{mv_\perp^2}{2} - eV_0 + W\right)$$

1(a). take $v^2 = v^2 - \frac{2eV_0}{m}$

$$J = \left(\frac{m}{2\pi}\right)^3 \iint_{-\infty}^{\infty} d^2 v_\parallel \int_0^{\infty} f\left(\frac{mv_\parallel^2}{2} + \frac{mv_\perp^2}{2} + W\right) dv_\perp^2$$

The saturation current

$$I_s = 4\pi R^2 J = \frac{m^3R^2}{2\pi^2}\iint_{-\infty}^{\infty} d^2v_\parallel \int_0^\infty f\left(\frac{mv_\parallel^2}{2} + \frac{mv_\perp^2}{2} + W\right) dv_\perp^2$$

1(b). In the Boltzmann approximation

$$f\left(\frac{mv^2}{2} + W\right) \approx e^{-\beta\left(\frac{mv^2}{2}+W\right)}$$

$$J = \left(\frac{m}{2\pi}\right)^3 \iint_{-\infty}^{\infty} e^{-\beta\left(\frac{mv_\parallel^2}{2}\right)} d^2v_\parallel \int_0^\infty e^{-\beta\left(\frac{mv_\perp^2}{2}+W\right)} dv_\perp^2 = \frac{m^2}{2^2\pi^2}T\int_0^\infty e^{-\beta\left(\frac{mv_\perp^2}{2}+W\right)} dv_\perp^2 = \frac{m}{2\pi^2}T^2e^{-\frac{W}{T}}$$

$$I_s = 4\pi R^2 J = \frac{2mR^2}{\pi}T^2e^{-\frac{W}{T}}$$

2. In this situation the electron need energy that

$$\mu_{\text{anode}} + W' - \mu_{\text{cathode}} = W' + eV_{\text{battery}},$$

so we expect that the expression for the current will be:

$$I \propto T^2e^{-\beta\left(W'+eV_{\text{battery}}\right)}$$

2(a). In the photoelectric effect the cathode isn't heated so one assumes zero temperature Fermi occupation for the cathode also. That means there is no thermionic emission. The electrons are excited out of the cathode by photons with a fixed energy. These photons need to be with enough energy for the electrons to pass the energetic barrier, which is again $W' + eV_{\text{battery}}$. So if we measure the voltage needed to stop the current for a given photons energy, we know that at that voltage we have:

$$E_{\text{photon}} = W' + eV_{\text{battery}} \implies W' = E_{\text{photon}} - eV_{\text{battery}}$$

So we can calculate the anode work function.

3. We can treat the cathode as a capacitor. The charge will be $Q = VC$, and the current: $I = \frac{dQ}{dt} = C\frac{d}{dt}V$. Here $C = \frac{1}{R}$, and V are the capacitance and voltage of a charged sphere in vacuum with respect to infinity. On the other hand we know from item (2) that when there is a voltage difference the current is $I \sim I_0e^{-\beta eV}$, where I_0 is independent of V. So we need to solve:

$$\frac{d}{dt}V = \frac{I_0}{C}e^{-\beta eV}$$

$$e^{\beta eV}dV = \frac{I_0}{C}dt \implies \frac{T}{e}e^{\beta eV} = \frac{I_0}{C}t + \tilde{C}$$

$$V = \frac{T}{e}\ln\left(\frac{e}{T}\frac{I_0}{C}t + \frac{e}{T}\tilde{C}\right) \implies Q(t) = \frac{CT}{e}\ln\left(\frac{e}{T}\frac{I_0}{C}t + \frac{e}{T}\tilde{C}\right)$$

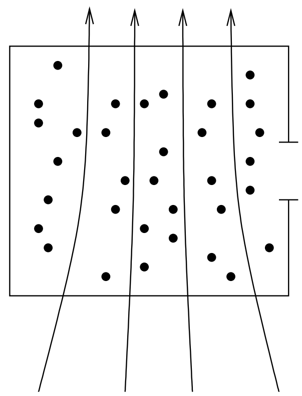
given that $Q(t=0) = 0$:

$$Q(t) = \frac{CT}{e}\ln\left(\frac{e}{T}\frac{I_0}{C}t + 1\right)$$

3(a). The result is limited because it is valid only when the Boltzmann approximation is valid, and it does not take into consideration finite number of electrons.

Exercise 6040 - Effusion of electrons from a box in magnetic field

A box with electrons of mass m is subjected to a magnetic field B. The single particle interaction is described by $-\gamma B\sigma_z$. The chemical potential of the electrons inside the box is μ. A hole through one of the walls is drilled. The electrons that are emitted from the hole with a velocity in the range $v < v' < v + dv$ are filtered, and subsequently their spin is measured. The measured current is defined as $I = I_\uparrow + I_\downarrow$.



1. Find the ratio $\alpha(B;\mu) = (I_\uparrow - I_\downarrow)/I$.
2. Find a linear approximation for $\alpha(B;\mu)$ regarded as a function of the magnetic field.
3. What is the maximal value of $\alpha(B;\mu)/B$, and what is the range for which the result is valid.

Answer

1. The flux for N electrons in volume V with velocity v is :

$$J = \iint_{|\theta|<\frac{\pi}{2}} \frac{d\Omega}{4\pi}\frac{N}{V} v\cos\theta = \frac{1}{4}\left(\frac{N}{V}\right) v$$

The number of the particles N:

$$N = \int g\left(E\right) f\left(E-\mu\right) dE = \int F\left(v\right) dv$$

The number of spin up/down electrons in the velocity range $v < v' < v + dv$ is given by :

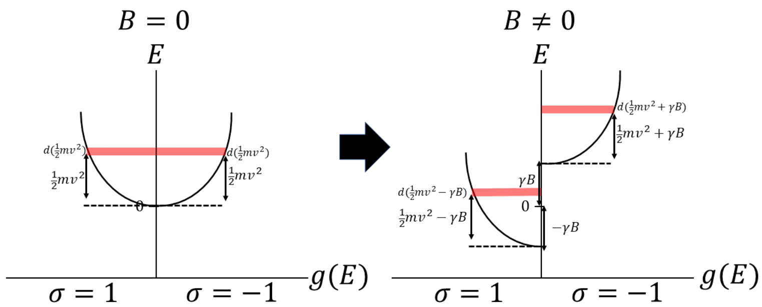
$$N_{\uparrow/\downarrow}(v < v' < v + dv) = dN_{\uparrow/\downarrow} = g\left(\frac{1}{2}mv^2 \mp \gamma B\right) f\left(\frac{1}{2}mv^2 \mp \gamma B - \mu\right) d\left(\frac{1}{2}mv^2 \mp \gamma B\right)$$

So

$$I_\uparrow = J_\uparrow dAdt = \frac{1}{4}\frac{g\left(\frac{1}{2}mv^2 - \gamma B\right) f\left(\frac{1}{2}mv^2 - \gamma B - \mu\right) d\left(\frac{1}{2}mv^2 - \gamma B\right)}{V} vdAdt$$

$$I_\downarrow = J_\downarrow dAdt = \frac{1}{4}\frac{g\left(\frac{1}{2}mv^2 + \gamma B\right) f\left(\frac{1}{2}mv^2 + \gamma B - \mu\right) d\left(\frac{1}{2}mv^2 + \gamma B\right)}{V} vdAdt$$

But we can see that $g\left(\frac{1}{2}mv^2 - \gamma B\right) = g\left(\frac{1}{2}mv^2 + \gamma B\right) = g\left(\frac{1}{2}mv^2\right)$ and
$d\left(\frac{1}{2}mv^2 - \gamma B\right) = d\left(\frac{1}{2}mv^2 + \gamma B\right) = d\left(\frac{1}{2}mv^2\right)$



So we get

$$\alpha(B;\mu) = \frac{I_\uparrow - I_\downarrow}{I_\uparrow + I_\downarrow} = \frac{f\left(\frac{1}{2}mv^2 - \gamma B - \mu\right) - f\left(\frac{1}{2}mv^2 + \gamma B - \mu\right)}{f\left(\frac{1}{2}mv^2 - \gamma B - \mu\right) + f\left(\frac{1}{2}mv^2 + \gamma B - \mu\right)}$$

2. The linear approximation for $\alpha(B;\mu)$ regarded as a function of the magnetic field:

$$f\left(\frac{1}{2}mv^2 \pm \gamma B - \mu\right) \approx f\left(\frac{1}{2}mv^2 - \mu\right) \pm f'\left(\frac{1}{2}mv^2 - \mu\right)\gamma B$$

$$\alpha(B;\mu) \approx \frac{-f'\left(\frac{1}{2}mv^2 - \mu\right)}{f\left(\frac{1}{2}mv^2 - \mu\right)}\gamma B$$

$$f'\left(\frac{1}{2}mv^2 - \mu\right) = \left(\frac{1}{e^{\beta\left(\frac{1}{2}mv^2 - \mu\right)} + 1}\right)' = -\frac{\beta e^{\beta\left(\frac{1}{2}mv^2 - \mu\right)}}{\left(e^{\beta\left(\frac{1}{2}mv^2 - \mu\right)} + 1\right)^2} = -f\left(\frac{1}{2}mv^2 - \mu\right)^2 \beta e^{\beta\left(\frac{1}{2}mv^2 - \mu\right)}$$

$$\alpha(B;\mu) \approx f\left(\frac{1}{2}mv^2 - \mu\right) e^{\beta\left(\frac{1}{2}mv^2 - \mu\right)}\beta\gamma B = \frac{1}{1 + e^{-\beta\left(\frac{1}{2}mv^2 - \mu\right)}}\beta\gamma B$$

3. For

$$\beta\left(\frac{1}{2}mv^2 - \mu\right) \gg 1$$

$$\frac{mv^2}{2} \gg (T + \mu)$$

$$\alpha(B;\mu)/B \to \frac{\gamma}{T}$$