# Statistical Mechanics - Class Exercise 11

January 19, 2023

## Exercise 8481 - Mass on a spring

A balance for measuring weight consists of a sensitive spring which hangs from a fixed point. The spring constant is $K$. The balance is at temperature $T$ and gravity acceleration is $g$ in the $x$ direction. A small mass $m$ hangs at the end of the spring. There is an option to apply an external force $F(t)$, to which $x$ is conjugate or apply an external vector potential $A(t)$.

1. Find the partition function $Z$.
2. Find $\langle x\rangle$ and $\langle x^2\rangle$ and $\mathrm{Var}(x)$.
3. Write a Langevin equation for $x(t)$, with friction $\eta$, and a random force $f(t)$.
4. Assuming $\langle f(t)f(0)\rangle = C\delta(t)$, find $\mathrm{Var}(x)$, and deduce what is $C$ by comparing with the canonical result.
5. Assuming $x$ is measured in the lab by averaging over time period $t_0$, what is the minimal mass that can be meaningfully measured?
6. Describe the external force $F(t)$ by a scalar potential and demonstrate FDT.
7. Describe the external force $F(t)$ by a vector potential and demonstrate FDT.

Note: $\int \frac{d\omega}{(\omega^2-\omega_0^2)^2+\gamma^2\omega^2} = \frac{\pi}{\gamma\omega_0^2}$.

## Answer

1. The Hamiltonian of the system is:

$$H = \frac{p^2}{2m} + \frac{1}{2}Kx^2 - mgx$$

We can rewrite the Hamiltonian in the following way

$$\begin{aligned} H =& \frac{p^2}{2m} + \frac{1}{2}K\left(x^2 - 2\frac{mg}{K}x + \left(\frac{mg}{K}\right)^2\right) - \frac{(mg)^2}{2K} \\ =& \frac{p^2}{2m} + \frac{1}{2}K\left(x - x_0\right)^2 - \frac{(mg)^2}{2K}, x_0 = \frac{mg}{K} \end{aligned}$$

So, the partition function

$$\frac{e^{\frac{\beta(mg)^2}{2K}}}{\lambda_T}\int_{-\infty}^{\infty} e^{-\beta\frac{K}{2}(x-x_0)^2}dx = \frac{e^{\frac{\beta(mg)^2}{2K}}}{\lambda_T}\sqrt{\frac{2\pi}{\beta K}}$$

2. The Gaussian in the partition function is centered around $x_0$, therefore we deduce that

$$\langle x \rangle = \frac{1}{Z_x} \int_{-\infty}^{\infty} x e^{-\beta \frac{K}{2}(x-x_0)^2} dx = \frac{1}{Z_x} \int_{-\infty}^{\infty} (x + x_0)\, e^{-\beta \frac{K}{2} x^2} dx = x_0$$

In order to find $\langle x^2 \rangle$ we use the equipartition and the Virial theorems

$$\langle x \cdot \frac{\partial U}{\partial x} \rangle = \langle p \cdot \frac{\partial \mathcal{K}}{\partial p} \rangle = \left\langle \frac{p^2}{m} \right\rangle = T$$

$$\langle x K (x - x_0) \rangle = T$$

$$\langle x^2 \rangle = \frac{T}{K} + x_0^2$$

And the variance of $x$ is

$$\text{Var}(x) = \langle x^2 \rangle - \langle x \rangle^2 = \frac{T}{K}$$

3. The Langevin equation for this system is:

$$\dot{x} = \frac{\partial H}{\partial p}, \dot{p} = -\frac{\partial H}{\partial x}$$

$$m\ddot{x} + \eta \dot{x} + Kx - mg = f(t)$$

4. First we change the variable $x \to x - x_0$, that in equilibrium $\langle x \rangle = 0$ ,the Langevin equation became

$$m\ddot{x} + \eta \dot{x} + Kx = f(t)$$

After Fourier transform

$$(-m\omega^2 - i\eta\omega + K) x_\omega = f_\omega$$

Multiply by the conjugate the both sides and averege

$$\left( \left(K - m\omega^2\right)^2 + \eta^2 \omega^2 \right) \left\langle |x_\omega|^2 \right\rangle = \left\langle |f_\omega|^2 \right\rangle$$

Using the Wiener-Khinchin theorem $\left\langle |f_\omega|^2 \right\rangle = \tilde{C}_{ff}(\omega) \times t$, we get

$$\tilde{C}_{xx}(\omega) = \frac{1}{m^2} \frac{\tilde{C}_{ff}(\omega)}{(\omega^2 - \frac{K}{m})^2 + (\gamma\omega)^2}, \gamma = \frac{\eta}{m}$$

The Fourier transform of $\delta(t)$ is 1, so the force-correlation is $\tilde{C}_{ff}(\omega) = C$. We get

$$\text{Var}(x) = C_{xx}(t = 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{C}_{xx}(\omega) d\omega = \frac{1}{2\pi} \frac{C}{m^2} \int_{-\infty}^{\infty} \frac{d\omega}{(\omega^2 - \frac{K}{m})^2 + (\gamma\omega)^2} = \frac{C}{2\eta K}$$

when we use $\int \frac{d\omega}{(\omega^2 - \omega_0^2)^2 + \gamma^2 \omega^2} = \frac{\pi}{\gamma \omega_0^2}$
so we get

$$\text{Var}(x) = \frac{C}{2\eta K} = \frac{T}{K}$$

$$C = 2\eta T = \nu$$

Finally we get

$$\tilde{C}_{xx}(\omega) = \frac{T}{m} \frac{2\gamma}{(\omega^2 - \frac{K}{m})^2 + (\gamma\omega)^2}$$

5. We measure $x(t)$ in the lab and average over time period $t_0$. This measurement introduces a new random variable $X$

$$X = \frac{1}{t_0} \int_0^{t_0} x(t) dt$$

This variable has a mean value $\langle X \rangle$ and variance $\mathrm{Var}(X)$. In order to get a meaningful measurement, it has to obey the condition - $\langle X \rangle \gg \sqrt{\mathrm{Var}(X)}$. From this condition, the minimal mass $m_{\min}$ will be found. The mean value of $X$ is

$$\langle X \rangle = \frac{1}{t_0} \int_0^{t_0} \langle x(t) \rangle \, dt = x_0$$

The variance (after we change variables $x \to x - x_0$)

$$\mathrm{Var}(X) = \frac{1}{t_0^2} \int_0^{t_0} dt' \int_0^{t_0} dt'' \langle x(t') x(t'') \rangle = \frac{1}{t_0} \tilde{C}_{xx}(\omega = 0) = \frac{2\eta T}{t_0 K^2}$$

so, the minimal mass is given by

$$x_0 = \frac{mg}{K} \gg \sqrt{\frac{2\eta T}{t_0 K^2}}$$

$$m \gg \sqrt{\frac{2\eta T}{g^2 t_0}} = m_{\min}$$

6. The force $F(t)$ is described by a scalar potential $U$ so the interaction term is $-\varepsilon\,(t)\,x$, so the conjugate variables are $x$ and $\varepsilon$.
Averaging the Langevin formula

$$m\langle \ddot{x} \rangle + \eta \langle \dot{x} \rangle + K \langle x \rangle = \varepsilon$$

After Fourier transforming

$$x_\omega = \frac{1}{(-m\omega^2 + K) - i\omega\eta} \epsilon_\omega = \chi_\omega \varepsilon_\omega$$

Hence, we get the correlation function from FDT

$$\tilde{C}_{xx}(\omega) = \frac{2T}{\omega} \mathrm{Im}\chi_\omega = \frac{T}{m} \frac{2\gamma}{(\omega^2 - \frac{K}{m})^2 + (\gamma\omega)^2}$$

7. The force $F(t)$ is described by a vector potential $A(t)$, the interaction term is $v \cdot A$, so the conjugate variables are $v$ and $-A$. The averaged Langevin formula becomes

$$m\langle \dot{v} \rangle + \eta \langle v \rangle + K \langle x \rangle = -\dot{A}$$

After Fourier transforming

$$v_\omega = \frac{1}{\frac{i}{\omega}(-m\omega^2 + K) + \eta} i\omega A_\omega$$

$$v_\omega = \frac{1}{m}\frac{\omega^2}{(-\omega^2+\frac{K}{m})-i\omega\gamma}A_\omega$$

We use that $F(t) = -\dot{A} \rightarrow F_\omega = i\omega A_\omega$
The correlation function is

$$v_\omega = \chi_\omega A_\omega$$

$$\tilde{C}_{vv}(\omega) = \frac{2T}{\omega}\mathrm{Im}\chi_\omega = \frac{T}{m}\frac{2\gamma\omega^2}{(\omega^2-\frac{K}{m})^2+(\gamma\omega)^2} = \omega^2\tilde{C}_{xx}(\omega)$$

what we can get immediately from

$$\tilde{C}_{vv}(\omega) = \frac{1}{t}\left\langle |v_\omega|^2\right\rangle = \frac{1}{t}\omega^2\left\langle |x_\omega|^2\right\rangle = \omega^2\tilde{C}_{xx}(\omega)$$

## Exercise 8483 - Millikan experiment

Consider a Millikan-type experiment whose purpose is to measure the charge $e$ of a particle with mass $m$. The particle is located between plates of capacitor, where the electric field $\mathcal{E}$ is in the "up" direction, while the gravitation $g$ is in the "down" direction. The distance between the plates is $L$, and the temperature of the system is $T$. Due to the poor vacuum the particle executes a Brownian motion that is described by a Langevin equation with friction force $-\eta v$. The charge of the electron is estimated via $\delta F = e\mathcal{E} - mg = 0$. In item (1) the system is prepared with a single particle in the middle. In item (3) assume a uniform gas of $N$ particles. In both cases the current is integrated during a time interval $t$, and the charge $Q = \int I(t')dt'$ is inspected as "readout".

1. Assuming that $\delta F = 0$, determine the time $t_d$ such that for $t \ll t_d$ it is not likely to get charge readout.
2. What is the $\delta F$ for which the condition $t \ll t_d$ is no longer valid. We shall regard this value, call it $\delta_1$, as the resolution of the measurement.
3. Assuming that $\delta F = 0$, determine the power spectrum $C_{II}(\omega)$ of the current $I(t)$.
4. Assume that the time of the measurement is $t$. What is the $\delta F$ for which the condition $\langle Q\rangle \ll \sqrt{\mathrm{var}(Q)}$ is no longer valid. We shall regard this value, call it $\delta_N$, as the resolution of the measurement.
5. Express the ratio $\delta_N/\delta_1$ as a function of $N$ and $t/t_d$.

**Tips:** In the absence of fluctuations $\delta F = 0$ is indicated by having zero readout. In item (3) the "readout" is a current versus voltage ("IV") measurement, and $\delta F = 0$ is indicated by zero current. Due to the fluctuations there is some blurring which determines the resolution $\delta_N$. In order to calculate the fluctuations in item (3) define the one-particle current as the velocity (up to a prefactor).

## Answer

1. The Langevin equation for the Brownian motion:

$$m\dot{v} + \eta v = f(t)$$

with $\langle f(t)\rangle = 0$ so for steady state $\langle v\rangle = 0$.
Solving this equation for the spreading of the particle yields $\left\langle (x(t)-x(0))^2\right\rangle = 2Dt$ where $D = \frac{T}{\eta}$. It follows, that it would be unlikely to get a charge readout for:

$$\sqrt{\frac{T}{\eta}\cdot t} \ll L \longrightarrow t_d = \frac{\eta L^2}{T}$$

2.When $e\mathcal{E} - mg \neq 0$ a "drift" term is to be added to the Langevin equation:

$$m\dot{v} + \eta v = f(t) + \delta F$$

So now the average velocity is $\langle v \rangle = \frac{\delta F}{\eta}$. In this case a minimum measurment time is $t = \frac{L}{\langle v \rangle}$ . But we would also want this time to be shorter than the spreading time $t_d$ we found in the previous item. This leads to the condition:

$$\frac{L}{\langle v \rangle} < t < \frac{\eta L^2}{T}$$

$$\delta F > \frac{T}{L} \equiv \delta_1$$

3. The current of a single particle is $I^1 = \frac{e}{L} v$. The power spectrum can be expressed as:

$$\left\langle |I^1_\omega|^2 \right\rangle = \left(\frac{e}{L}\right)^2 \left\langle |v_\omega|^2 \right\rangle$$

For Wiener-Khinchin theorem

$$C_{I^1 I^1}(\omega) = \left(\frac{e}{L}\right)^2 C_{vv}(\omega)$$

The Langevin equation:

$$(\eta - i\omega m)\, v_\omega = f_\omega$$

$$\left\langle |v_\omega|^2 \right\rangle = \frac{1}{\eta^2 + m^2\omega^2} \left\langle |f_\omega|^2 \right\rangle$$

$$C_{vv}(\omega) = \frac{C_{ff}(\omega)}{\eta^2 + m^2\omega^2}$$

when, for white noise $C_{ff}(\omega) = \nu = 2\eta T$, so

$$C_{I^1 I^1}(\omega) = \left(\frac{e}{L}\right)^2 \frac{T}{m} \frac{2\gamma}{\gamma^2 + \omega^2}, \gamma = \frac{\eta}{m}$$

The total current is a sum over single particle currents and so the power of the total current will be $N$ times the power from a single particle:

$$C_{II}(\omega) = N \left(\frac{e}{L}\right)^2 \frac{T}{m} \frac{2\gamma}{\omega^2 + \gamma^2}$$

4. The readout is the total charge $Q = \int_0^t I(t')dt'$. We can know that that the readout is not because the drift power only for $\sqrt{\mathrm{var}(Q)} \gg \langle Q \rangle$.

$$\langle Q \rangle = \langle I \rangle t = N\frac{e}{L} \langle v \rangle t = N\frac{e}{L}\frac{\delta F}{\eta} t$$

in the other side (that calculated assuming $\delta F=0$):

$$\mathrm{Var}(Q)=\left\langle Q^{2}\right\rangle =\int_{0}^{t}\int_{0}^{t}dt'dt''\left\langle I(t')I(t'')\right\rangle =C_{II}\left(\omega=0\right)t=N\left(\frac{e}{L}\right)^{2}\frac{2T}{\eta}t$$

The condition on $\delta F$ is then:

$$\sqrt{N\left(\frac{e}{L}\right)^{2}\frac{2T}{\eta}t}\gg N\frac{e}{L}\frac{\delta F}{\eta}t$$

and this break for

$$\delta F>\sqrt{\frac{T\eta}{Nt}}\equiv\delta_{N}$$

5. The ratio $\frac{\delta_N}{\delta_1}$ can be expressed as a function of $N$ and $t/t_d$:

$$\frac{\delta_{N}}{\delta_{1}}=\frac{1}{\sqrt{N\frac{t}{t_{d}}}}$$

## Exercise 8490 - Stochastic rate equation

Consider $N$ classical particles in a two site system. The two sites are subjected to a potential difference $\varepsilon$. The temperature of the system is $T$. Define $n\in[-N,N]$ as the occupation difference. In items (3-6) assume that the thermalization process can be described by a stochastic rate equation

$$\frac{dn}{dt}=-\gamma n+A(t)$$

where $A(t)$ is a noisy term that reflects the fluctuations of the potential difference. Assuming that it has an average value $A_0$ and a power spectrum $\phi(\omega)$, it follows that $n$ relaxes to an average value $\langle n\rangle$, with fluctuations that are characterized by a power spectrum $C(\omega)$.

1. Write what is the interaction energy $H_{\text{int}}$ of $n$ with the field $\varepsilon$. Later you will have to be careful with the identification of the conjugate variables.
2. Using the canonical formalism find what are $\langle n\rangle$ and $\mathrm{Var}(n)$. Additionally provide approximations for small $\varepsilon$.
3. Determined what is $A_0$ such that $\langle n\rangle$ would be consistent with the canonical result. Assuming small $\varepsilon$ deduce that $A_0\propto\varepsilon$, and find the pre-factor.
4. What is the $\chi(\omega)$ that characterizes the response of $n$ to the applied potential in the linear-response regime? Assume that the dynamics is described by the stochastic rate equation; care to identify correctly the conjugate variables; and take into account your answer to item (3).
5. Deduce from the fluctuation-dissipation relation what is the power spectrum $C(\omega)$. Care to use the appropriate definition for $\chi(\omega)$, else the result will come out wrong.
6. Deduce what is the power spectrum $\phi(\omega)$ that is required in order to reproduce $C(\omega)$ from the stochastic rate equation.

**Advice:** In item (5) verify that your result is consistent with the answer to item (2). Likewise you can debug the numerical pre-factor in your answer to item (6). Care about factors of “2” in your answers. Failure to provide strictly correct pre-factors will be regarded as an essential error.

## Answer

1. We take the potential difference $\varepsilon$ in That way, potential of $-\frac{\varepsilon}{2}$ in site 1and $\frac{\varepsilon}{2}$ in site 2

$$\mathcal{H}_{\text{int}} = -\frac{\varepsilon}{2}N_1 + \frac{\varepsilon}{2}N_2 = -\frac{\varepsilon}{2}n$$

2. The partition function is

$$Z_1 = e^{\beta\frac{\varepsilon}{2}} + e^{-\beta\frac{\varepsilon}{2}} = 2\cosh\left(\beta\frac{\varepsilon}{2}\right)$$

$$Z = (Z_1)^N = 2^N \cosh^N\left(\beta\frac{\varepsilon}{2}\right)$$

From this we get

$$\langle n \rangle = \frac{1}{\beta}\frac{\partial \ln(Z)}{\partial\frac{\varepsilon}{2}} = N\tanh\left(\beta\frac{\varepsilon}{2}\right)$$

We notice that in the limit $\varepsilon \to 0$ we have $\langle n \rangle \to 0$ and in the limit $\varepsilon \to \infty$ we have $\langle n \rangle \to N$, as expected.

$$\text{Var}(n) = \frac{1}{\beta^2}\frac{\partial^2 \ln(Z)}{\partial\frac{\varepsilon}{2}^2} = \frac{N}{\cosh^2\left(\beta\frac{\varepsilon}{2}\right)}$$

If we approximate for small $\varepsilon$ we get

$$\langle n \rangle \approx \frac{N}{T}\frac{\varepsilon}{2}$$

$$\text{Var}(n) \approx N\left(1 - \left(\frac{1}{T}\frac{\varepsilon}{2}\right)^2\right)$$

3. The stochastic rate equation

$$\frac{dn}{dt} = -\gamma n + A(t)$$

after averaging we get in steady state

$$\langle n \rangle = \frac{A_0}{\gamma}$$

we require

$$\frac{A_0}{\gamma} = \frac{N}{T}\frac{\varepsilon}{2}$$

$$\to A_0 = \gamma\frac{N}{T}\frac{\varepsilon}{2}$$

4. By taking the Fourier transform of the averaging stochastic rate equation

$$n_\omega = \frac{A_{0\omega}}{(\gamma - i\omega)}$$

$$n_\omega = \frac{\gamma N\beta}{(\gamma - i\omega)}\frac{\varepsilon_\omega}{2}$$

$$\to \chi(\omega) = \frac{\gamma N\beta}{(\gamma - i\omega)}$$

5.

$$\chi(\omega) = \frac{\gamma N\beta}{(\gamma - i\omega)} = \frac{\gamma^2 N\beta}{(\gamma^2+\omega^2)} + i\frac{\gamma N\beta\omega}{(\gamma^2+\omega^2)}$$

From FDT we get

$$\mathrm{Im}\chi(\omega) = \tanh\left(\frac{\omega}{2T}\right) C_{nn}(\omega)$$

in the classical limit $\omega \to 0$

$$\frac{\mathrm{Im}\chi(\omega)}{\omega} = \frac{1}{2T} C_{nn}(\omega) = \frac{\gamma N\beta}{\gamma^2+\omega^2}$$

$$C_{nn}(\omega) = N\frac{2\gamma}{\gamma^2+\omega^2}$$

6.

$$\left\langle |n_\omega|^2 \right\rangle = \frac{\left\langle |A_\omega|^2 \right\rangle}{\gamma^2+\omega^2}$$

$$C_{nn}(\omega) = \frac{\phi(\omega)}{\gamma^2+\omega^2}$$

$$\phi(\omega) = N2\gamma$$

## Exercise 8034 - Brownian particle on a ring

The motion of a classical Brownian particle on a 1D ring is described by the Langevin equation $m\ddot{\theta} + \eta\dot{\theta} = f(t)$, where $f(t)$ is due to a noisy electromotive force that has a correlation function $\langle f(t')f(t'')\rangle = C_f(t'-t'')$. The power spectrum $\tilde{C}_f(\omega)$ is defined as the Fourier transform of the correlation function. We consider two cases:

1. High temperature white noise $\tilde{C}_f(\omega) = \nu$.
2. Zero temperature noise $\tilde{C}_f(\omega) = c|\omega|$.

We define the angular velocity of the particle as $v = \dot{\theta}$, and its Cartesian coordinate as $x = \sin(\theta)$. In the absence of noise the dynamics is characterized by the damping time $t_c = m/\eta$.
In items (3)-(5) you should assume a spreading scenario: the particle is initially ($t = 0$) located at $\theta \sim 0$. The spreading during the transient period $0 < t < t_c$ is assumed to be negligible. In item (6) assume that the particle had been launched in the far past ($t = -\infty$): accordingly there is no preferred location on the ring.

1. Find the exact correlation function $\langle v(t)v(0)\rangle$ in case (a).
2. Find the correlation function $\langle v(t)v(0)\rangle$ for $t \gg t_c$ in case (b).
3. Find the spreading $S(t) \equiv \langle \theta(t)^2\rangle$ for $t \gg t_c$ in case (a).
4. Find the spreading $S(t) \equiv \langle \theta(t)^2\rangle$ for $t \gg t_c$ in case (b).
5. Express $\langle x(t)^2\rangle$ for a spreading scenario given $S(t)$.
6. Express the correlation function $\langle x(t)x(0)\rangle$ given $S(t)$.

7. Write the explicit long time expression for $\langle x(t)x(0)\rangle$ in case (b), and deduce what is the critical value $\eta_c$ above which a "phase transition" is expected in the response characteristics of the system.

Tips: For a Gaussian variable that has zero average $\langle \exp i\varphi\rangle = \exp[-(1/2)\langle\varphi^2\rangle]$.
The Fourier transform of $|\omega|$ has zero area, with negative tails $-1/(\pi t^2)$.
If you fail to solve (6), assume that the answer is the same as in (5), and proceed to (7).

## Answer

1. We will start with writing the Langevin equation for the velocity $m\dot{v} + \eta v = f(t)$, we can solve it with Fourier transform:

$$(-i\omega m + \eta)v_\omega = f(\omega)$$

$$v_\omega = \frac{1}{m}\frac{f(\omega)}{\gamma - i\omega}, \gamma = \frac{\eta}{m}$$

Now we can take square absolute value from both sides and average :

$$\langle |v_\omega|^2\rangle = \frac{1}{m^2}\frac{\langle |f(\omega)|^2\rangle}{\gamma^2 + \omega^2}$$

From the Wiener-Khinchin theorem we get that $\langle |f(\omega)|^2\rangle = \tilde{C}_f(\omega) \times t$, so we get:

$$\tilde{C}_v(\omega) = \frac{1}{m^2}\frac{\tilde{C}_f(\omega)}{\gamma^2 + \omega^2}$$

For case (a) $\tilde{C}_f(\omega) = \nu$ we get:

$$\tilde{C}_v(\omega) = \frac{1}{m^2}\frac{\nu}{\gamma^2 + \omega^2}$$

After inverse Fourier transform:

$$C_v(t) = \int \frac{d\omega'}{2\pi}\frac{1}{m^2}\frac{\nu}{\gamma^2 + \omega'^2}e^{-i\omega' t} = \frac{\nu}{2m^2\gamma}\int d\omega \frac{1}{\pi}\frac{\gamma}{\gamma^2 + \omega'^2}e^{-i\omega' t}$$

This is a Lorentzian, so we get:

$$C_v(t) = \langle v(t)v(0)\rangle = \frac{\nu}{2m^2\gamma}e^{-\gamma|t|} = \frac{\nu}{2m^2\gamma}e^{-\frac{|t|}{t_c}}$$

2. Now we use the same equation but in case (b) $\tilde{C}_f(\omega) = c|\omega|$:

$$\tilde{C}_v(\omega) = \frac{1}{m^2}\frac{\tilde{C}_f(\omega)}{\gamma^2 + \omega^2} = \frac{1}{m^2}\frac{c|\omega|}{\gamma^2 + \omega^2}$$

we need to do inverse Fourier transform:

$$C_v(t) = \frac{c}{m^2}\int \frac{d\omega'}{2\pi}\frac{|\omega'|}{\gamma^2 + \omega'^2}e^{-i\omega' t}$$

We know that the change of $|\omega|$ is slow except near the $\omega = 0$, so the shape of $\omega \approx 0$ is determined by the higher $t$ and the shape of $\omega \gg 0$ is determined by the lower $t$. We take the limit $t \gg t_c$ so we can neglect $\omega > \frac{1}{t_c}$ and get:

$$C_v(t) = \langle v(t)v(0)\rangle = \frac{c}{m^2}\int \frac{d\omega'}{2\pi}\frac{|\omega'|}{\gamma^2+\omega'^2}e^{-i\omega' t} = \frac{c}{\eta^2}\int \frac{d\omega'}{2\pi}\frac{|\omega'|}{1+t_c^2\omega'^2}e^{-i\omega' t} \approx \frac{c}{\eta^2}\int \frac{d\omega'}{2\pi}|\omega'|e^{-i\omega' t}$$

We know that the Fourier transform of $|\omega|$ has zero area, with negative tails $-\frac{1}{\pi t^2}$, so we get:

$$\begin{aligned}\int \frac{d\omega'}{2\pi}|\omega'|e^{-i\omega' t} &= -\int_{-\infty}^{0}\frac{d\omega'}{2\pi}\omega' e^{-i\omega' t} + \int_{0}^{\infty}\frac{d\omega'}{2\pi}\omega' e^{-i\omega' t}\\ &= i\partial_t \lim_{\eta\to 0}\int_{-\infty}^{0}\frac{d\omega'}{2\pi}e^{-i\omega' t+\eta\omega'} + i\partial_t \lim_{\eta\to 0}\int_{0}^{\infty}\frac{d\omega'}{2\pi}e^{-i\omega' t-\eta\omega'}\\ &= -i\partial_t \lim_{\eta\to 0}\frac{1}{2\pi}\frac{1}{-it+\eta} - i\partial_t \lim_{\eta\to 0}\frac{1}{2\pi}\frac{1}{-it-\eta} = \frac{1}{\pi}\partial_t\frac{1}{t} = -\frac{1}{\pi t^2}\end{aligned}$$

$$C_v(t) = \langle v(t)v(0)\rangle \approx \frac{c}{\eta^2}\int \frac{d\omega'}{2\pi}|\omega'|e^{-i\omega' t} = -\frac{c}{\eta^2\pi t^2}$$

3. In the beginning we define $\dot{\theta} = v$, so we get:

$$\theta(t) = \int_0^t dt' v(t')$$

$$\theta^2(t) = \int_0^t\int_0^t dt'dt'' v(t')v(t'')$$

$$\langle\theta^2(t)\rangle = \int_0^t\int_0^t dt'dt''\langle v(t')v(t'')\rangle = \int_0^t\int_0^t dt'dt'' C_v(t'-t'')$$

We can see that $t', t''$ are independent variables, so we can choose that $t' > t''$ and double the result. We can do a change of variables to two dependent variables $T = t' \to 0 < T < t, \tau = t' - t'' \to 0 < \tau < T$.

$$\langle\theta^2(t)\rangle = 2\int_0^t dT\int_0^T d\tau C_v(\tau) = 2\int_0^t dT\int_0^T d\tau\frac{\nu}{2m\eta}e^{-\frac{|\tau|}{\left(\frac{m}{\eta}\right)}}$$

The correlation decay very fast so in the limit $t \gg t_c$ we can take the integral to infinite:

$$\langle\theta^2(t)\rangle = 2\int_0^t dT\int_0^\infty d\tau\frac{\nu}{2m\eta}e^{-\frac{|\tau|}{\left(\frac{m}{\eta}\right)}} = 2t\frac{\nu}{2m\eta}\left(\frac{m}{\eta}\right) = \frac{\nu}{\eta^2}t$$

Or, in the short way

$$\langle\theta^2(t)\rangle = \int_0^t\int_0^t dt'dt''\langle v(t')v(t'')\rangle = \int_0^t\int_0^t dt'dt'' C_v(t'-t'') = \tilde{C}_v(\omega=0)\cdot t = \frac{\nu}{\eta^2}t$$

4. In the same way:

$$\langle\theta^2(t)\rangle = 2\int_0^t dT\int_0^T d\tau C_v(\tau)$$

We can neglect the spreading during the transient period $0 < t < t_c$, so we take the limit:

$$\langle\theta^2(t)\rangle = 2\int_{t_c}^{t} dT \int_0^T d\tau C_v(\tau)$$

The solution we found to $C_v(t)$ in case (b) is good just for $t \gg t_c$, so we need to divide the integral to two parts (we assume that the limit $T = t_c$ is the lower limit to our solution):

$$\langle\theta^2(t)\rangle = 2\int_{t_c}^{t} dT \left(\int_0^\infty d\tau C_v(\tau) - \int_T^\infty d\tau C_v(\tau)\right)$$

The part $\int_0^\infty d\tau C_v(\tau) = \tilde{C}_v(\omega = 0) = 0$, so we get:

$$\langle\theta^2(t)\rangle = -2\int_{t_c}^{t} dT \int_T^\infty d\tau C_v(\tau) = 2\int_{t_c}^{t} dT \int_T^\infty d\tau \frac{c}{\eta^2\pi\tau^2} = 2\int_{t_c}^{t} dT \frac{c}{\eta^2\pi T} = \frac{2c}{\pi\eta^2}\ln\frac{|t|}{t_c}$$

5. We defined $x = \sin\theta$:

$$\langle x^2(t)\rangle = \langle\sin^2\theta(t)\rangle = \langle\frac{\left(e^{i\theta} - e^{-i\theta}\right)^2}{-4}\rangle = \frac{1}{4}\langle\left(2 - e^{i2\theta} - e^{-i2\theta}\right)\rangle = \frac{1}{2} - \frac{1}{4}\langle e^{i2\theta}\rangle - \frac{1}{4}\langle e^{-i2\theta}\rangle$$

We get a tip that for a Gaussian variable that has zero average $\langle e^{i\varphi}\rangle = e^{-(1/2)\langle\varphi^2\rangle}$. Because $\theta$ is a Gaussian variable and it has zero average we get:

$$\langle x^2(t)\rangle = \frac{1}{2} - \frac{1}{4}\langle e^{i2\theta}\rangle - \frac{1}{4}\langle e^{-i2\theta}\rangle = \frac{1}{2} - \frac{1}{4}e^{-2\langle\theta^2\rangle -} - \frac{1}{4}e^{-2\langle\theta^2\rangle} = \frac{1}{2}\left(1 - e^{-2\langle\theta^2\rangle}\right) = \frac{1}{2}\left(1 - e^{-2S(t)}\right)$$

Note that $\langle(2\theta)^2\rangle = \langle(-2\theta)^2\rangle = 4\langle\theta^2\rangle$

6. In the previous sections we assumed that $\theta(0) \approx 0$ and we talked about short times, so we could treat $\theta$ like a coordinate and calculate $\langle\theta(t)^2\rangle$. Now there isn't a preferred location on the ring so we can't calculate $S(t) = \langle\theta(t)^2\rangle$, because $\theta$ is not well defined. So we can't calculate $\langle x^2(t)\rangle$ like before, just the correlation between two different times $\langle x(t)x(0)\rangle$.

By definition:

$$\langle x(t)x(0)\rangle = \int_0^{2\pi}\int_0^{2\pi} \sin(\theta_t)\sin(\theta_0)\rho(\theta_t, \theta_0)d\theta_t d\theta_0$$

The formula for conditional probability is:

$$\rho(A|B) = \frac{\rho(A, B)}{\rho(B)} \rightarrow \rho(A, B) = \rho(A|B)\rho(B)$$

$$\langle x(t)x(0)\rangle = \int_0^{2\pi}\int_0^{2\pi} \sin(\theta_t)\sin(\theta_0)\rho(\theta_t|\theta_0)\rho(\theta_0)d\theta_t d\theta_0$$

We get that in $t = 0$, $\theta_0$ has a Uniform distribution, namely $\rho(\theta_0) = \frac{1}{2\pi}$.

Additionally $\rho(\theta_t|\theta_0)$ is the probability to find $\theta_t$ when we know where is $\theta_0$, and this is like the previous section, when we assumed that $\theta_0 = 0$.

The probability $\rho(\theta_t|\theta_0)$ depends only on the difference between $\theta_t$ and $\theta_0$, it doesn't depends on one of them, so let's define $\delta\theta = \theta_t - \theta_0$, when $\rho(\theta_t|\theta_0)d\theta_t = \rho(\delta\theta)d\delta\theta$

By using a trigonometric identities we get:

$$\int_0^{2\pi}\int_0^{2\pi}\sin(\theta_t)\sin(\theta_0)\rho(\theta_t|\theta_0)\rho(\theta_0)d\theta_t d\theta_0 = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi}\left(\cos(\delta\theta)-\cos(2\theta_0+\delta\theta)\right)\rho(\delta\theta)d\delta\theta d\theta_0 =$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi}\cos(\delta\theta)\rho(\delta\theta)d\delta\theta d\theta_0 - \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi}\cos(2\theta_0+\delta\theta)\rho(\delta\theta)d\delta\theta d\theta_0 =$$

The first integral doesn't depend on $\theta_0$, the integral on $\theta_0$ in the second term give 0, and we get:

$$= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi}\cos(\delta\theta)\rho(\delta\theta)d\delta\theta d\theta_0 = \frac{1}{2}\int_0^{2\pi}\cos(\delta\theta)\rho(\delta\theta)d\delta\theta = \frac{1}{2}\langle\cos(\delta\theta)\rangle$$

When we take $\theta_0 = 0$ we get that $\delta\theta$ is the same $\theta$ we define in the previous section and get:

$$\frac{1}{2}\langle\cos(\delta\theta)\rangle = \frac{1}{4}\langle e^{i\delta\theta}+e^{-i\delta\theta}\rangle = \frac{1}{4}\left(\langle e^{i\delta\theta}\rangle + \langle e^{-i\delta\theta}\rangle\right) = \frac{1}{2}e^{-\frac{1}{2}\langle\delta\theta^2\rangle} = \frac{1}{2}e^{-\frac{1}{2}S(t)}$$

7. For case (b):

$$S(t) = \frac{2c}{\pi\eta^2}\ln\frac{t}{t_c}$$

$$\langle x(t)x(0)\rangle = \frac{1}{2}e^{-\frac{1}{2}S(t)} = \frac{1}{2}\left(\frac{t}{t_c}\right)^{-\frac{c}{\pi\eta^2}}$$

From the FDT we get the relationship between the correlation function and the response:

$$\text{Im}\chi \sim \frac{\omega}{2T}\tilde{C}_{xx}(\omega)$$

In the DC limit, we get:

$$\text{Im}\chi = \frac{\omega}{2T}\tilde{C}_{xx}(\omega=0)$$

When:

$$\tilde{C}_{xx}(\omega=0) = \int_{-\infty}^{\infty}C_{xx}(t)dt = \int_{-\infty}^{\infty}\frac{1}{2}\left(\frac{t}{t_c}\right)^{-\frac{c}{\pi\eta^2}}dt$$

We define “phase transition” as When the response diverges. this will happen when $\frac{c}{\pi\eta^2}\leq 1$, so we get:

$$\eta_c = \sqrt{\frac{c}{\pi}}$$