HOME EX. 1: FOURIER RECAP

We define the Fourier transform of an a-preiodic function $f(x) = f(x+a)$ by:

$$\begin{aligned} f(x) &= \sum_k f(k) e^{ikx} \\ f(k) &= \frac{1}{a}\int_{-a/2}^{a/2} f(x) e^{-ikx} dx \end{aligned} \tag{1}$$

If the function $f(x)$ has infinite period, then it can be defined as follows:

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(k) e^{ikx} dk \\ f(k) &= \int_{\infty}^{\infty} f(x) e^{-ikx} dx \end{aligned} \tag{2}$$

For the following questions you may use any of these definitions.

1.
What is the fourier transform $U(k)$ of the function:

$$U(x) = U_0 a \sum_n [2\delta(x - na) - \delta(x - (n + 1/4)a) - \delta(x - (n + 3/4)a)] \tag{3}$$

With $n \in [-\infty, .., -1, 0, .., \infty]$

(Taken from Exam 2011 MOED A)

2.
Define $f(x) = x$ for $x \in [-\frac{a}{2}, \frac{a}{2}]$.
Find a fourier series: $\tilde{f}(x) = \sum_k \tilde{f}_k e^{ikx}$, so that $\tilde{f}(x) = f(x)$ for $x \in [-a/2, a/2]$ (that is: they agree also at $x = \pm a/2$: $\tilde{f}(\pm a/2) = f(\pm a/2)$).
Note: If f(x) is not continous in x_0, then its fourier representation will give $\frac{1}{2}[f(x_0-) + f(x_0+)]$ in $x = x_0$.

3.
Dirac's comb is defined by:

$$\Delta(x) = \sum_{n=-\infty}^{\infty} \delta(x - an) \tag{4}$$

a) Find its Fourier transform Δ_k and plot schematically $V(k) = |\Delta_k|^2$ as a function of k.
We now consider a finite comb:

$$\Delta_N(x) = \sum_{n=-N/2}^{N/2} \delta(x - an), \qquad (N \text{ integer}) \tag{5}$$

b) Write the Fourier transform of $\Delta_N(x)$.
c) For a given N, what is the maximum value of $V_N(k) = |\Delta_N(k)|^2$. For what values of k does V_N attain this value ?
d) Plot $V(k)_N/N^2$ as a function of k for different N (you can use a computer program), describe the difference between the finite and infinite comb.

Diffraction in the the far field approximation is given by the Fourier transform of the potential (here $\Delta(x)$). The difference between items (a) and (c) are the expected difference in the intensity between an infinite crystal and finite one.