Solution:

a) The wavefunction is given by

$$|\psi\rangle = C\left(+4i\left|\phi_{2,1,0}\right\rangle - 2\sqrt{2}\left|\phi_{2,2,-1}\right\rangle + \left|\phi_{1,0,0}\right\rangle\right)$$

Where $\phi_{n,l,m}$ is a normalized eigenfunction of the H-atom with the corresponding quantum numbers. Due to the orthonormality of the eigenfunction we find that

$$\langle\psi|\psi\rangle = C^2(16+8+1) = 25C^2 \Rightarrow C = \frac{1}{5}.$$

b) The expectation value of the energy is given by:

$$\begin{aligned}\langle E\rangle = \left\langle\psi\middle|\hat{H}\middle|\psi\right\rangle &= C^2\left(-4i\langle\phi_{2,1,0}| - 2\sqrt{2}\langle\phi_{2,2,-1}| + \langle\phi_{1,0,0}|\right)\hat{H}\left(+4i\left|\phi_{2,1,0}\right\rangle\right. \\ &\left.- 2\sqrt{2}\left|\phi_{2,2,-1}\right\rangle + \left|\phi_{1,0,0}\right\rangle\right) \\ &= C^2 16\left\langle\phi_{2,1,0}\middle|\hat{H}\middle|\phi_{2,1,0}\right\rangle + C^2 8\left\langle\phi_{2,2,-1}\middle|\hat{H}\middle|\phi_{2,2,-1}\right\rangle \\ &+ C^2\left\langle\phi_{1,0,0}\middle|\hat{H}\middle|\phi_{1,0,0}\right\rangle = \frac{1}{25}(16E_2 + 8E_2 + E_1) = \frac{1}{25}\left(24\frac{E_1}{4} + E_1\right) \\ &= \frac{7}{25}E_1\end{aligned}$$

c) The expectation value of the squared angular momentum is:

$$\begin{aligned}\langle L^2\rangle = \left\langle\psi\middle|\widehat{L^2}\middle|\psi\right\rangle & \\ &= C^2\left(-4i\langle\phi_{2,1,0}| - 2\sqrt{2}\langle\phi_{2,2,-1}| + \langle\phi_{1,0,0}|\right)\widehat{L^2}\left(+4i\left|\phi_{2,1,0}\right\rangle\right. \\ &\left.- 2\sqrt{2}\left|\phi_{2,2,-1}\right\rangle + \left|\phi_{1,0,0}\right\rangle\right) \\ &= C^2 16\left\langle\phi_{2,1,0}\middle|\widehat{L^2}\middle|\phi_{2,1,0}\right\rangle + C^2 8\left\langle\phi_{2,2,-1}\middle|\widehat{L^2}\middle|\phi_{2,2,-1}\right\rangle \\ &+ C^2\left\langle\phi_{1,0,0}\middle|\widehat{L^2}\middle|\phi_{1,0,0}\right\rangle \\ &= \frac{1}{25}\left(16\hbar^2 1(1+1) + 8\hbar^2 2(2+1) + \hbar^2 0(0+1)\right) \\ &= \frac{\hbar^2}{25}(32+48) = 3\frac{1}{5}\hbar^2\end{aligned}$$

d) The expectation value of the z-component of the angular momentum is:

$$\begin{aligned}\langle L_z\rangle = \left\langle\psi\middle|\widehat{L_z}\middle|\psi\right\rangle & \\ &= C^2\left(-4i\langle\phi_{2,1,0}| - 2\sqrt{2}\langle\phi_{2,2,-1}| + \langle\phi_{1,0,0}|\right)\widehat{L_z}\left(+4i\left|\phi_{2,1,0}\right\rangle\right. \\ &\left.- 2\sqrt{2}\left|\phi_{2,2,-1}\right\rangle + \left|\phi_{1,0,0}\right\rangle\right) \\ &= C^2 16\left\langle\phi_{2,1,0}\middle|\widehat{L_z}\middle|\phi_{2,1,0}\right\rangle + C^2 8\left\langle\phi_{2,2,-1}\middle|\widehat{L_z}\middle|\phi_{2,2,-1}\right\rangle \\ &+ C^2\left\langle\phi_{1,0,0}\middle|\widehat{L_z}\middle|\phi_{1,0,0}\right\rangle = \frac{1}{25}(16\hbar 0 + 8\hbar(-1) + \hbar 0) = -\frac{8}{25}\hbar\end{aligned}$$

e) The probability of finding the atom in the state ϕ_{210} is given by the square of the absolute value of the projection of the state on the ϕ_{210} state. Namely,

$$P(2,1,0) = \left|\left\langle\phi_{2,1,0}\middle|\psi\right\rangle\right|^2 = C^2|4i|^2 = \frac{16}{25}.$$

f) The lowest eigenfunction of the H-atom is $\phi_{1,0,0} = \frac{2}{a_0^{3/2}} e^{-\frac{r}{a_0}}$. Therefore, the probability density of finding the electron at a distance r from the nucleus is given by:

$$p(r) = \frac{r^2 4}{a_0^3} e^{-\frac{2r}{a_0}}$$

In order to find the most likely distance we have to find the maximum of the probability density. Therefore, we take its derivative with respect to r.

$$\frac{dp(r)}{dr} = \frac{8r}{a_0^3} e^{-\frac{2r}{a_0}} - \frac{r^2 8}{a_0^4} e^{-\frac{2r}{a_0}}$$

Requesting that the derivative is zero in order to find the maximum we find:

$$\frac{8r}{a_0^3} e^{-\frac{2r}{a_0}} - \frac{r^2 8}{a_0^4} e^{-\frac{2r}{a_0}} = 0 \Rightarrow r^{\#} = a_0$$

Where $r^{\#}$ denotes the most likely distance. Note that there is a small difference between the reduced Bohr radius, and the Bohr radius due to the use of the reduced mass rather than the electron mass in the definition.

g) At $t = 0$, the z-component of the angular momentum is measured and it is found that $L_z = 0$, which implies that the wavefunction collapses to:

$$\psi(\vec{r}, t = 0) = \tilde{c}[\phi_{100}(\vec{r}) + 4i\phi_{210}(\vec{r})]$$

At a later time, the wave function is given by:

$$\psi(\vec{r}, t) = \tilde{c}\left[\phi_{100}(\vec{r}) \mathrm{e}^{-i\frac{\mathrm{E_1 t}}{\hbar}} + 4i\phi_{210}(\vec{r}) \mathrm{e}^{-i\frac{\mathrm{E_2 t}}{\hbar}}\right]$$

Were $\tilde{c}$ is the new normalization coefficient, and with a similar calculation to the one we have already done before we get $\tilde{c} = \frac{1}{\sqrt{17}}$.

The probability to measure the 2nd energy is given by $P(E_2) = \sum_{l,m} |\langle 2, l, m|\psi\rangle|^2$, and in our case we get $P(E_2) = |\tilde{c}4i|^2 = \frac{16}{17}$.