Gravity 1 - Tutorial 1

Euclidean and Spherical Geometries

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1 Euclidean Geometry

1.1 The Construction Axioms

In Euclid's *Elements* (300 BC), he assumes that certain constructions can be done and he states these assumptions in a list called his *axioms* (or postulates). In the first three axioms he assumes that it is possible to:

1. Draw a straight line segment between any two points.
2. Extend a straight line segment indefinitely.
3. Draw a circle with given center and radius.

Axioms 1 and 2 say we have a *straightedge*, an instrument for drawing arbitrarily long line segments. Today we replace them by the single axiom that a *line* can be drawn through any two points. The straightedge (unlike a ruler) has no scale marked on it and hence can be used only for drawing lines - not for measurement. Euclid separates these two functions by giving measurement functionality only to the *compass* - the instrument assumed in Axiom 3.

There is no better way to start our journey than with Euclid's first proposition.

Exercise: Construct an equilateral triangle on a given finite straight line AB.

Solution: (see Figure 1)

1. Draw the circle with center A and radius AB.
2. Draw the circle with center B and radius AB.
3. Draw the line segments from A and B to the intersection C of the two circles just constructed.

$AB = AC$ because they are both radii of the first circle. $AB = BC$ because they are both radii of the second circle. Therefore $AB = AC = BC$.

(Homework exercise: Bisect a line segment).

1.2 The Parallel Axiom

The fourth and the fifth Axioms are:

4. All right angles equal one another.

5. If a straight line crossing two straight lines makes the interior angles on one side together less than two right angles, then the two straight lines, if

Figure 1: Construction of Euclid's first proposition. Picture from Moses Mendelssohn's (1729-1786) museum, Berlin.

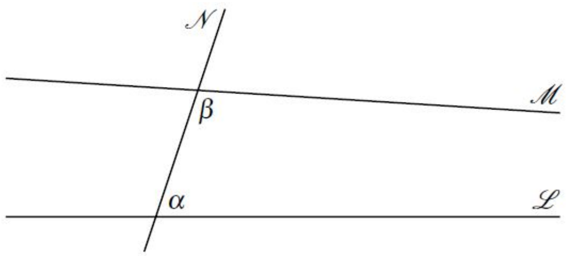

Figure 2: Non parallel lines

produced indefinitely, will meet on that side.

It follows that if $\mathscr{L}$ and $\mathscr{M}$ do not meet on either side, then $\alpha + \beta = \pi$.

At some time mathematicians were beginning to speculate whether the fifth postulate was indeed necessary. It might well be a consequence of the first four postulates. Over the years, along with many attempts to prove the fifth postulate from the other four, mathematicians came up with several formulations equivalent to the fifth postulate in the presence of the other four. Here are only a few:

- For any line and a point not on the line, there is exactly one line through the point that does not meet the line (Playfair's axiom).
- The sum of the angles in every triangle is π (triangle postulate).
- There exists a pair of similar, but not congruent, triangles.
- In a right-angled triangle, the square of the hypotenuse equals the sum of the squares of the other two sides (Pythagoras Theorem).
- The Law of cosines, a generalization of Pythagoras' Theorem.

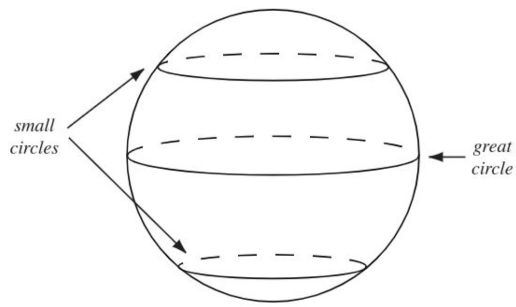

Figure 3: Circles on a sphere

- There is no upper limit to the area of a triangle. (Wallis axiom).

(Homework exercise: prove fifth axiom $\Rightarrow$ triangle postulate)

2 Spherical Geometry

In hindsight, the oldest non-Euclidean geometry human investigated is the two dimensional geometry on a sphere. To start with, we consider a sphere of radius R in three dimensional Euclidean space, centered at the origin. A *point* on a sphere is just a point of the space that lies on the sphere. A *circle* on the sphere is the intersection of the sphere with a plane (for example, all longitudes and latitudes). A *line* on the sphere is the intersection of the sphere with a plane that goes through the center (for example, all longitudes but only the equator latitude). Therefore, lines on the sphere are the *great circles*, the circles with maximal circumference. A *polygon* is a circuit of line segments and vertices. Notice that there are spherical triangles, quadrilaterals, but also digons and monogons which cannot exist in the plane!

Exercise: Which Euclidean axioms hold on a sphere? Elaborate.

Solution:

1. A line segment can be drawn between any two points, but it is not unique. Between two "generic" points one can draw two line segments of the same great circle going though the two points. That is not so bad, since it is still one line. But - between two *antipodal points* (like north and south poles) there are infinite number of lines going through them.

2. No, since after $2\pi R$ the line repeats itself.

3. No, the hemisphere (a monogon) has a radius of $\frac{\pi}{2}R$.

4. True.

5. The parallel axiom does not hold. Any two (non overlapping) lines meet at exactly two antipodal points. Playfair's axiom would read: for any line and a point not on the line, there is **no** line through the point that does not meet the line. In other words, there are no parallel lines.

2.1 Spherical Law of Cosines

Since spherical geometry is non-Euclidean, the usual planar law of cosine does not hold for spherical triangles. Nonetheless, there is a more general spherical law of cosines[1]. A side of a spherical triangle is an arc of a great circle, and for a unit sphere this arc length equals the corresponding central angle. An angle on the sphere is the angle between the two planes making the corresponding lines. The *spherical law of cosines* on a unit sphere reads

Spherical law of cosines

$$\cos(c) = \cos(a)\cos(b) + \sin(a)\sin(b)\cos(\gamma) \tag{1}$$

See Figure 4. (Homework exercise: prove the spherical law of cosines)

Exercise: *i*. Show that the planar law of cosines is the leading approximation of (1) for a small spherical triangle. *ii*. Write the spherical Pythagoras theorem.

Solution: *i*. For $a, b, c \ll 1$ approximate (1) by $\cos x \approx 1 - x^2$, $\sin x \approx x$

$$1 - \frac{1}{2}c^2 \approx \left(1 - \frac{1}{2}a^2\right)\left(1 - \frac{1}{2}b^2\right) + ab\cos\gamma \approx 1 - \frac{1}{2}a^2 - \frac{1}{2}b^2 + ab\cos\gamma \tag{2}$$

$$\Downarrow$$

$$c^2 \approx a^2 + b^2 - 2ab\cos\gamma \tag{3}$$

ii. Pythagoras theorem is the special case where $\gamma = \frac{\pi}{2}$. Therefore the spherical Pythagoras theorem is

$$\cos(c) = \cos(a)\cos(b) \tag{4}$$

Exercise: Compute the angles of a quadrilateral triangle with sides: *i*. $a = c = b = \frac{\pi}{2}$. *ii*. $a = b = c = \frac{\pi}{4}$.

[1]and also a spherical law of sines.

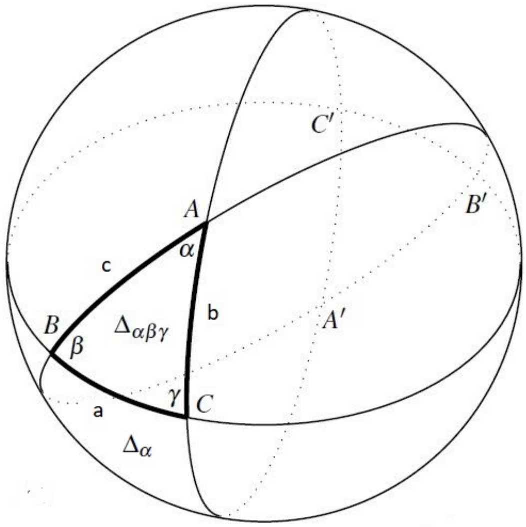

Figure 4: Spherical triangle

Solution: From symmetry all the angles are equal. *i.* Just plug in (1). $0 = 0 + 1\cos\gamma \Rightarrow \gamma = \frac{\pi}{2}$. *ii.* $\frac{1}{\sqrt{2}} = \frac{1}{2} + \frac{1}{2}\cos\gamma \Rightarrow \cos\gamma = \sqrt{2} - 1 \Rightarrow \gamma = 65.53°$.

2.2 Harriot's Theorem

Consider a sphere with radius R. The sum of the angles of a spherical triangle is greater then π. The *angular excess* of a triangle with angles α, β, γ is $\mathcal{E}_{\alpha\beta\gamma} = \alpha + \beta + \gamma - \pi$. Thomas Harriot's[2] theorem (1603) states that the angular excess is proportional to the triangle area $\Delta_{\alpha\beta\gamma}$where

Harriot's theorem

$$\mathcal{E}_{\alpha\beta\gamma} = \kappa \Delta_{\alpha\beta\gamma} \tag{5}$$

where the proportionality constant, with dimensions L^{-2}, is $\kappa = \frac{1}{R^2}$. κ is the (Gaussian) *curvature* of the sphere.

Remarks:

- For a triangle much smaller then the scale of the sphere, the angular excess goes to zero $\mathcal{E}_{\alpha\beta\gamma} = \frac{\Delta_{\alpha\beta\gamma}}{R^2} \to 0$, as the triangle becomes flat and Euclidean.

[2]Thomas Harriot (1560-1621) was an English astronomer, mathematician, ethnographer and translator to whom the theory of refraction is attributed. He made contributions in navigational techniques, sometimes credited with the introduction of the potato to the British Isles, and was the first person to make a drawing of the Moon through a telescope, on 5 August 1609, about four months before Galileo Galilei.

- From (5) we can interpret the curvature of the sphere $\kappa = \frac{\mathcal{E}_{\alpha\beta\gamma}}{\cdot_{\alpha\beta\gamma}}$ as an angular excess density.

- Two similar triangles have the same angles, and the same angular excess. By Harriot's theorem they have also the same area, so they are also congruent. This reflects the breaking of the fifth Euclidean axiom, there does not exist a pair of similar, but not congruent, triangles.

Exercise: Prove Harriot's theorem.

Solution: See Figure 4. Denote the total surface area of the sphere as $S = 4\pi R^2$. The angle α makes slice the sphere into two opposite lunes (digons) with areas $\Delta_{\alpha\beta\gamma} + \Delta_\alpha$. The same hold also for β and γ. Therefore the total area of the sphere is

$$S = 2\left(\Delta_{\alpha\beta\gamma} + \Delta_\alpha + \Delta_\beta + \Delta_\gamma\right). \tag{6}$$

The equations for the angles come from the fact that the proportion of the area of a lune with angle α to the total area of the sphere equals the proportion of α to the total angle of 2π. Likewise for β and γ. Therefore

$$\begin{aligned} \frac{\alpha}{2\pi} S &= \Delta_{\alpha\beta\gamma} + \Delta_\alpha \\ \frac{\beta}{2\pi} S &= \Delta_{\alpha\beta\gamma} + \Delta_\beta \\ \frac{\gamma}{2\pi} S &= \Delta_{\alpha\beta\gamma} + \Delta_\gamma. \end{aligned} \tag{7}$$

Summing equations (7) yields

$$\frac{1}{2\pi}\left(\alpha + \beta + \gamma\right) S = 3\Delta_{\alpha\beta\gamma} + \Delta_\alpha + \Delta_\beta + \Delta_\gamma. \tag{8}$$

Plugin (6)

$$\frac{1}{2\pi}\left(\alpha + \beta + \gamma\right) S = 2\Delta_{\alpha\beta\gamma} + \frac{S}{2} \tag{9}$$

and arranging

$$\left(\alpha + \beta + \gamma - \pi\right) S = 4\pi\Delta_{\alpha\beta\gamma} \tag{10}$$

and using $S = 4\pi R^2$, $\mathcal{E}_{\alpha\beta\gamma} = \alpha + \beta + \gamma - \pi$, $\kappa = \frac{1}{R^2}$,

$$\mathcal{E}_{\alpha\beta\gamma} R^2 = \Delta_{\alpha\beta\gamma} \quad \Rightarrow \quad \mathcal{E}_{\alpha\beta\gamma} = \kappa\Delta_{\alpha\beta\gamma} \tag{11}$$

Exercise: Verify Harriot's theorem for the triangle with sides $a = b = c = \frac{\pi}{2}R$.

Solution: We saw that this triangle has angles $\alpha = \beta = \gamma = \frac{\pi}{2}$. The sides of this triangle are the intersections of the xy, xz, yz planes with the sphere. These planes divide the sphere into eight equal triangles like this. Therefore, the angular excess is $\mathcal{E}_{\alpha\beta\gamma} = \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} - \pi = \frac{\pi}{2}$, and the triangle area is $\Delta_{\alpha\beta\gamma} = \frac{S}{8} = \frac{4\pi R^2}{8} = \frac{\pi R^2}{2}$. Indeed, $\mathcal{E}_{\alpha\beta\gamma} = \frac{1}{R^2}\Delta_{\alpha\beta\gamma}$.

(Homework exercise: verify for icosahedron and dodecahedron tessellations of the sphere)

2.3 Circumference of a Circle

Exercise: *i*. Find the circumference of a circle $C(r)$ with spherical radius r, on a sphere with radius R. *ii*. Approximate for $\frac{r}{R} \ll 1$. Show that at zeroth order $C(r)$ reduces to the Euclidean expression, and express the curvature κ by the leading order of $C(r)$ in $\frac{r}{R}$.

Solution: *i*. For convenience (and without loss of generality) we choose the center of the circle at the north pole (see Figure 3). The angle to the z-axis is

$$\theta = \frac{r}{R} \tag{12}$$

where r is the arc length from the pole to the circle along a longitude, i.e. the radius of the circle. We calculate the circumference of the circle in the horizontal plane it lies in. The planar radius is the distance from the z-axis ρ, given by

$$\rho = R\sin\theta = R\sin\left(\frac{r}{R}\right) \tag{13}$$

Therefore

$$\boxed{C(r) = 2\pi\rho = 2\pi R\sin\left(\frac{r}{R}\right)} \tag{14}$$

Circumference of a spherical circle

ii. For $\frac{r}{R} \ll 1$

$$C(r) = 2\pi R\sin\left(\frac{r}{R}\right) = 2\pi R\left(\frac{r}{R} - \frac{1}{3!}\left(\frac{r}{R}\right)^3 + \mathcal{O}\left(\left(\frac{r}{R}\right)^5\right)\right) \tag{15}$$

At zeroth order we get the Euclidean expression

$$C^{(0)}(r) = 2\pi r \tag{16}$$

At second order

$$C^{(2)}\left(r\right)=2\pi r-\frac{\pi}{3}\frac{r^{3}}{R^{2}}$$

The curvature is

$$\kappa=\frac{1}{R^{2}}=\frac{3}{\pi}\left(\frac{2\pi r-C^{(2)}\left(r\right)}{r^{3}}\right) \tag{17}$$

Curvature as the deviation of the circumference of a small circle

(Homework exercise: same with circle area)