Gravity 1 - Tutorial 2

The Euclidean Plane

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1 Groups and Geometry

In 1872, the German mathematician Felix Klein initiated the so-called *Erlangen program* (under the name of his university), a method of characterizing geometries based on group theory. He suggested that **geometry is the study of invariants of groups of transformations**.

A *group* is a set G, equipped with a binary operation $\cdot : G \times G \to G$ such that the following axioms are satisfied:

- Associativity: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in G$.
- Identity element: There exists an element $e \in G$ such that $a \cdot e = e \cdot a = a$ for all $a \in G$.
- Inverse element: For any $a \in G$ there is an (inverse) element $b \in G$, such that $a \cdot b = b \cdot a = e$.

If a group is also commutative, i.e. $a \cdot b = b \cdot a$ for all $a, b \in G$, it is called an *Abelien group*.

Algebraic examples: The integers with operation of addition; the rational numbers without zero with operation of multiplication; the unit complex numbers with operation of multiplication; vectors with operation of vector addition.

1.1 The Euclidean Isometry Group

We **define** the Euclidean *quadrance* (distance squared) between two points in the plane $\mathbb{R}^2$, $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ as

Euclidean quadrance

$$q(P_1P_2) = (x_1 - x_2)^2 + (y_1 - y_2)^2 \tag{1}$$

A transformation of the plane is an invertible function $f : \mathbb{R}^2 \to \mathbb{R}^2$, i.e., a function that sends points to points. A transformation f is called an *isometry* (from the Greek for "same length") if it sends **any** two points, P_1 and P_2, to points $f(P_1)$ and $f(P_2)$ the same distance apart. In other words, an isometry preserves lengths, i.e., length is an *invariant* of isometry. It means that a Euclidean isometry preserves (1)

$$q(f(P_1)\, f(P_2)) = q(P_1P_2) \tag{2}$$

for any $P_1, P_2 \in \mathbb{R}^2$. The isometry group of the Euclidean plane is called the *Euclidean group* $E(2)$. Composition of two isometries f, g is an isometry $q\left(g\left(f\left(P_1\right)\right)g\left(f\left(P_2\right)\right)\right) = q\left(f\left(P_1\right)f\left(P_2\right)\right) = q\left(P_1P_2\right)$, and they satisfy the group axioms.

Exercise: Show that the set of points equidistant from two points is a line.

Solution: Let the two points be $P_1 = (a_1, b_1)$ and $P_2 = (a_2, b_2)$. If a point $P = (x, y)$ is equidistant from them then

$$q\left(P_1P\right) = q\left(P_2P\right) \tag{3}$$

$$\left(x - a_1\right)^2 + \left(y - b_1\right)^2 = \left(x - a_2\right)^2 + \left(y - b_2\right)^2 \tag{4}$$

$$x^2 - 2a_1x + a_1^2 + y^2 - 2b_1y + b_1^2 = x^2 - 2a_2x + a_2^2 + y^2 - 2b_2y + b_2^2 \tag{5}$$

$$2\left(a_2 - a_1\right)x + 2\left(b_2 - b_1\right)y + a_1^2 - a_2^2 + b_1^2 - b_2^2 = 0 \tag{6}$$

This is a linear equation, thus the points $P = (x, y)$ equidistant from P_1 and P_2 form a line.

An isometry should send the equidistant points from P_1 and P_2 to equidistant points from $f\left(P_1\right)$ and $f\left(P_2\right)$, which also form a line. It follows that an isometry sends lines to lines. Also, it should send parallel lines to parallel lines, otherwise some points at some nonzero distance would coincide after the transformation. A transformation that preserves lines and parallels is called an affine transformation. The affine group consists of combinations of linear transformations and translations $f\left(\boldsymbol{x}\right) = A\boldsymbol{x} + \boldsymbol{b}$. Thus, the Euclidean group is the subgroup of the affine group that also preserves lengths. Translations $t_{a,b}\left(x, y\right) = (x + a, y + b)$ are obviously isometries, since they preserve (1).

We now focus on finding the linear isometries. Linear transformations fix the origin, so we look for linear transformations that preserve the distance from the origin to any point $P = (x, y)$,

$$q\left(P\right) = x^2 + y^2 \tag{7}$$

1.2 The Orthogonal Group $O\left(2\right)$

In the realm of linear algebra, points are represented by vectors, thus the quadrance of a vector $\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ is

$$q\left(\boldsymbol{v}\right) = \left|\boldsymbol{v}\right|^2 = \boldsymbol{v}^T I \boldsymbol{v} = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y^2 \tag{8}$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the symmetric matrix that represents the quadratic form q. q induces an *inner product* between any two vectors $\boldsymbol{v} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \boldsymbol{u} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

$$g(\boldsymbol{v}, \boldsymbol{u}) = \frac{1}{2}\left(q(\boldsymbol{v} + \boldsymbol{u}) - q(\boldsymbol{v}) - q(\boldsymbol{u})\right) = \boldsymbol{v}^T I \boldsymbol{u} = x_1 x_2 + y_1 y_2 \tag{9}$$

The inner product also defines angles, so isometry also preserves angles. The quadrance of a vector is recovered by computing the inner product of the vector with itself

$$q(\boldsymbol{v}) = g(\boldsymbol{v}, \boldsymbol{v}) \tag{10}$$

Vectors $\boldsymbol{v}$ and $\boldsymbol{u}$ are called *orthogonal* if their inner product is zero

$$g(\boldsymbol{v}, \boldsymbol{u}) = 0 \tag{11}$$

Geometrically, two orthogonal vectors $\boldsymbol{v}$ and $\boldsymbol{u}$, are *perpendicular*. This is because if $g(\boldsymbol{v}, \boldsymbol{u}) = x_1 x_2 + y_1 y_2 = 0$, then the product of their slopes, $m_1 = \frac{y_1}{x_!}$ and $m_2 = \frac{y_2}{x_1}$, is $m_1 m_2 = -1$.[1]

Exercise:

1. Find the matrices representing the linear Euclidean isometries. This subgroup is denoted by $O(2)$. Show that the rows and columns of the matrix are orthonormal. Show that the subgroup with determinant $+1$ is equivalent to the multiplication group of unit complex numbers, $U(1)$.

2. Draw the group $O(2)$, i.e., draw the parameter space of the matrices you found. How many connected components does it have? Is it compact? Write the matrices of : identity transformation, rotations of $\frac{\pi}{2}, \pi, \frac{3\pi}{2}$ radians about the origin, reflections in the x-axis y-axis and $y = \pm x$ lines. Mark their points on the group drawing.

3. Show that the matrices of the subgroup with determinant $+1$ have no eigenvectors (except two matrices. Which ones?). These are *rotations.* Show that the matrices of the coset with determinant -1 have two independent eigenvectors, with eigenvalues ± 1. These are *reflections.*

[1]This is "obviously" true, and the origin for the name "orthogonal", which means "right angle". The point is that in the more general, modern sense, orthogonal and perpendicular are not synonyms. For a different inner product, algebraically orthogonal vectors are not geometrically perpendicular.

4. Use trigonometric functions to parameterize the group elements by two angles θ, α. Show that two successive rotations is a rotation of the sum of the angles. Show that two successive reflections is a rotation with twice the angle between the two lines of reflection.

5. Concentrate only on the rotation subgroup, $SO\left(2\right)$. Parameterize the group elements by *central projection* parameter β. Write the coordinate transformation $\beta\left(\theta\right)$ and find the "addition law" for β.

6. Show that the group law on the circle can be expressed as addition of sector areas. Consider also the following construction for the group law: given two points on the group, P and Q, draw a line connecting them. Now draw a parallel line that goes through the identity element. The "product" of P and Q is the second intersection of that parallel line with the circle. Show that this construction satisfies the group axioms, and that it yields the same group as the addition of angles.

Solution:

1.2.1 Orthogonal Matrices

• Isometries are transformations that preserve the quadrance (8) of any vector $\boldsymbol{v}$ (or the inner product (9) of any two vectors). Denote the linear operator by A, then the transformed vector is $A\boldsymbol{v}$, we demand that

$$\left(A\boldsymbol{v}\right)^{T} I \left(A\boldsymbol{v}\right) = \boldsymbol{v}^{T} I \boldsymbol{v} \tag{12}$$

thus

$$\boldsymbol{v}^{T}\left(A^{T} I A\right)\boldsymbol{v} = \boldsymbol{v}^{T} I \boldsymbol{v} \tag{13}$$

for **any** $\boldsymbol{v}$. Then the linear isometry R satisfies

$$A^{T} I A = I \tag{14}$$

$A \in O(2)$ is also called *orthogonal transformation*, represented by an *orthogonal matrix*. We solve (14) for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, b, c, d \in \mathbb{R}$.

$$A^T I A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} =$$
$$= \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \tag{15}$$

These are 3 equations with 4 unknowns

$$a^2 + c^2 = 1 \tag{16}$$
$$ab + cd = 0 \tag{17}$$
$$b^2 + d^2 = 1 \tag{18}$$

Isolate c from (17) and plug in (16)[2]

$$c = -\frac{ab}{d} \Rightarrow a^2 + \frac{a^2 b^2}{d^2} = a^2 \left(1 + \frac{b^2}{d^2}\right) = 1 \tag{19}$$

Manipulate (18) and substitute also

$$1 + \frac{b^2}{d^2} = \frac{1}{d^2} \Rightarrow \frac{a^2}{d^2} = 1 \tag{20}$$

so

$$d = \pm a \tag{21}$$

It follows that also

$$c = \mp b \tag{22}$$

and

$$a^2 + b^2 = 1 \tag{23}$$

Putting (21),(22),(23) together yields two types of orthogonal matrices $R, S \in O(2)$:

[2] should check $d = 0$ case at the end.

Orthogonal matrices

$$R = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \text{, } \det R = a^2 + b^2 = +1 \tag{24}$$

$$S = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \text{, } \det R = -\left(a^2 + b^2\right) = -1 \tag{25}$$

Of course from (14)

$$\det\left(A^T I A\right) = \det\left(A^T\right)\det(I)\det(A) = (\det(A))^2 = \det(I) = 1 \Rightarrow \det A = \pm 1 \tag{26}$$

The transformations with negative determinant flip the orientation of the space. What Klein is saying to us is that if we will restrict out isometry group only to the subgroup with positive determinant, the geometry will have another invariant geometric concept - orientation.

• We check that the rows and columns of R are orthonormal, as defined by the inner product (9).

First column squared

$$g\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix}\right) = a^2 + b^2 = 1 \ \checkmark \tag{27}$$

Second column squared

$$g\left(\begin{pmatrix} -b \\ a \end{pmatrix}, \begin{pmatrix} -b \\ a \end{pmatrix}\right) = (-b)^2 + a^2 = 1 \ \checkmark \tag{28}$$

First column times the second column

$$g\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} -b \\ a \end{pmatrix}\right) = a(-b) + ba = 0 \ \checkmark \tag{29}$$

The reader may check the rest.

• R can be written as

$$R = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = aI + bJ \tag{30}$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Since I is the identity matrix and

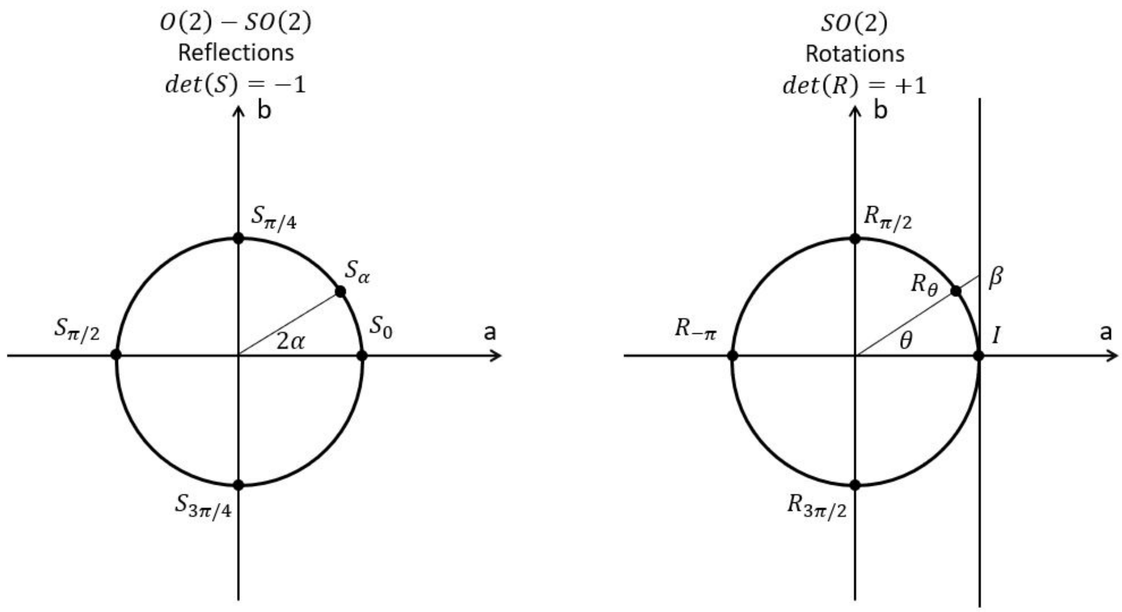

Figure 1: The Euclidean orthogonal group $O\left(2\right)$

$J^2 = -I$, the matrices of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ are equivalent to complex number $z = a+ib$, with addition and multiplication of matrices equivalent to the addition and multiplication of complex numbers. Notice also that geometrically J is a $\frac{\pi}{2}$ rotation matrix, like the action of i. The determinant $\det R = a^2 + b^2$ corresponds to the quadrance $|z| = \overline{z}z = a^2 + b^2$, so these orthogonal matrices correspond to unit complex numbers.

1.2.2 Drawing of the Group

We have two families of matrices with parameters $a, b \in \mathbb{R}^2$ with $a^2 + b^2 = 1$. These are two compact connected components of unit circles. Denote $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $R_{\frac{\pi}{2}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $R_\pi = -I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $R_{\frac{3\pi}{2}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $S_x = S_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $S_y = S_{\frac{\pi}{2}} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $S_{y=x} = S_{\frac{\pi}{4}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $S_{y=-x} = S_{\frac{3\pi}{4}} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. See Figure 1.

1.2.3 Rotations and Reflections

The characteristic polynomial of R is

$$\det(R-\lambda I)=\begin{vmatrix} a-\lambda & -b \\ b & a-\lambda \end{vmatrix}=(a-\lambda)^2+b^2=0 \tag{31}$$

It has solutions only if $a=\lambda$ and $b=0$. Therefore $\lambda=a=\pm 1$. These are the matrices I and R_π. Indeed, (Euclidean!) rotations have no eigenvectors, except for a rotation of 0 or π radians, with eigenvalues ± 1 respectively.

The characteristic polynomial of S is

$$\det(S-\lambda I)=\begin{vmatrix} a-\lambda & b \\ b & -a-\lambda \end{vmatrix}=\lambda^2-a^2-b^2=0 \tag{32}$$

$$\lambda^2=a^2+b^2=1 \;\Rightarrow\; \lambda=\pm 1 \;\forall a,b \tag{33}$$

S are reflections in lines through the origin, and the eigenvector with $\lambda=1$ is along the reflection axis, while the eigenvector with $\lambda=-1$ is perpendicular to the reflection axis.

1.2.4 Trigonometric Parameterization

We parameterize a point on each unit circle by an angle from from the a-axes. We denote by θ the angle in the rotations circle, and 2α the angle in the reflections circle. For R

$$a=\cos\theta \quad , \quad b=\sin\theta \tag{34}$$

$$R(\theta)=\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{35}$$

Rotation matrix by angle θ

For S

$$a=\cos 2\alpha \quad , \quad b=\sin 2\alpha \tag{36}$$

$$S(\alpha)=\begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} \tag{37}$$

Reflection matrix in a line with angle α

Of course $a^2+b^2=\cos^2\theta+\sin^2\theta=\cos^2(2\alpha)+\sin^2(2\alpha)=1$. The reason for choosing 2α as the angle in the circle of the reflection matrices, is that the matrix at a point on the circle of angle 2α acts as a reflection matrix in a line with angle α though the origin. We show that $\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix}$ is an eigenvector of $S(\alpha)$ with eigenvalue 1.

$$\begin{pmatrix}\cos 2\alpha & \sin 2\alpha\\ \sin 2\alpha & -\cos 2\alpha\end{pmatrix}\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix} = \begin{pmatrix}\cos 2\alpha\cos\alpha + \sin 2\alpha\sin\alpha\\ \sin 2\alpha\cos\alpha - \cos 2\alpha\sin\alpha\end{pmatrix}$$
$$= \begin{pmatrix}\left(\cos^2\alpha - \sin^2\alpha\right)\cos\alpha + 2\sin\alpha\cos\alpha\sin\alpha\\ 2\sin\alpha\cos\alpha\cos\alpha - \left(\cos^2\alpha - \sin^2\alpha\right)\sin\alpha\end{pmatrix} = \begin{pmatrix}\cos\alpha\left(\cos^2\alpha - \sin^2\alpha + 2\sin^2\alpha\right)\\ \sin\alpha\left(2\cos^2\alpha - \cos^2\alpha + \sin^2\alpha\right)\end{pmatrix}$$
$$= \begin{pmatrix}\cos\alpha\left(\cos^2\alpha + \sin^2\alpha\right)\\ \sin\alpha\left(\cos^2\alpha + \sin^2\alpha\right)\end{pmatrix} = \begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix} \quad (38)$$

Make a rotation by angle θ_1 followed by a rotation with angle θ_2,

$$R(\theta_2)R(\theta_1) = \begin{pmatrix}\cos\theta_2 & -\sin\theta_2\\ \sin\theta_2 & \cos\theta_2\end{pmatrix}\begin{pmatrix}\cos\theta_1 & -\sin\theta_1\\ \sin\theta_1 & \cos\theta_1\end{pmatrix}$$
$$= \begin{pmatrix}\cos\theta_2\cos\theta_1 - \sin\theta_2\sin\theta_1 & -\left(\cos\theta_2\sin\theta_1 + \sin\theta_2\cos\theta_1\right)\\ \sin\theta_2\cos\theta_1 + \cos\theta_2\sin\theta_1 & \cos\theta_2\cos\theta_1 - \sin\theta_2\sin\theta_1\end{pmatrix}$$
$$= \begin{pmatrix}\cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2)\\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2)\end{pmatrix} = R(\theta_1+\theta_2) \quad (39)$$

Different way: as with unit complex numbers, $R(\theta) = I\cos\theta + J\sin\theta = e^{J\theta}$, and $R(\theta_2)R(\theta_1) = e^{J\theta_2}e^{J\theta_1} = e^{J(\theta_1+\theta_2)} = R(\theta_1+\theta_2)$.

Make a reflection in a line with angle α_1 followed by a reflection in a line with angle α_2,

$$S(\alpha_2)S(\alpha_1) = \begin{pmatrix}\cos 2\alpha_2 & \sin 2\alpha_2\\ \sin 2\alpha_2 & -\cos 2\alpha_2\end{pmatrix}\begin{pmatrix}\cos 2\alpha_1 & \sin 2\alpha_1\\ \sin 2\alpha_1 & -\cos 2\alpha_1\end{pmatrix}$$
$$= \begin{pmatrix}\cos 2\alpha_2\cos 2\alpha_1 + \sin 2\alpha_2\sin 2\alpha_1 & -\left(\sin 2\alpha_2\cos 2\alpha_1 - \cos 2\alpha_2\sin 2\alpha_1\right)\\ \sin 2\alpha_2\cos 2\alpha_1 - \cos 2\alpha_2\sin 2\alpha_1 & \cos 2\alpha_2\cos 2\alpha_1 + \sin 2\alpha_2\sin 2\alpha_1\end{pmatrix}$$
$$= \begin{pmatrix}\cos(2(\alpha_2-\alpha_1)) & -\sin(2(\alpha_2-\alpha_1))\\ \sin(2(\alpha_2-\alpha_1)) & \cos(2(\alpha_2-\alpha_1))\end{pmatrix} = R(2(\alpha_2-\alpha_1)) \quad (40)$$

Notice that two reflections must make some rotation, since the product of two matrices with determinant -1 yields a matrix with determinant $+1$.

1.2.5 Central Projection Parameterization

Central projection of a circle means projecting the circle from its center onto a tangent line. We take the line to be the tangent to the identity element, i.e., to the point $(a,b) = (1,0)$. We parameterize the points on it by β, with $\beta = 0$ at $(1,0)$ and increasing upwards . If we send a ray from the center of the circle towards this line, the ray intersect the circle at one point and then intersect the line at one point (its projection), with some value of β. This identifies the projected point on the circle. Of course, this holds only for the right half of the circle, and we will leave it at that. Equivalently, β is the slope of the ray. Its range is $-\infty < \beta < \infty$.

We solve the equations of the circle and the ray

$$a^2 + b^2 = 1 \tag{41}$$

$$b = \beta a \tag{42}$$

$$a^2 + (\beta a)^2 = a^2 \left(1 + \beta^2\right) = 1 \tag{43}$$

Therefore

$$a = \frac{1}{\sqrt{1+\beta^2}} \tag{44}$$

$$b = \frac{\beta}{\sqrt{1+\beta^2}} \tag{45}$$

and the rotation matrices are

$$R(\beta) = \begin{pmatrix} \frac{1}{\sqrt{1+\beta^2}} & -\frac{\beta}{\sqrt{1+\beta^2}} \\ \frac{\beta}{\sqrt{1+\beta^2}} & \frac{1}{\sqrt{1+\beta^2}} \end{pmatrix} \tag{46}$$

Rotation matrix with central projection parameterization

The coordinate transformation between β and θ is

$$\beta = \frac{b}{a} = \frac{\sin\theta}{\cos\theta} = \tan\theta \tag{47}$$

Indeed, β is the slope of the ray[3]. Performing two rotations, with inclinations

[3] This is the origin for the name of the tangent function, it is the intersection of the ray with the tangent line.

of slopes β_1 and β_2, yields a rotation with inclination of slope β'

$$\beta' = \tan(\theta_1+\theta_2) = \frac{\sin(\theta_1+\theta_2)}{\cos(\theta_1+\theta_2)} = \frac{\sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2}{\cos\theta_2\cos\theta_1-\sin\theta_2\sin\theta_1} = \frac{\tan\theta_1+\tan\theta_2}{1-\tan\theta_1\tan\theta_2} \tag{48}$$

$$\beta' = \frac{\beta_1+\beta_2}{1-\beta_1\beta_2} \tag{49}$$

"Addition law" for slopes

1.2.6 Geometric Construction of the Group Law

• The area of a sector with angle $\Delta\theta$ in a circle of radius R is $A = \frac{\Delta\theta}{2\pi}\pi R^2 = \frac{1}{2}\Delta\theta R^2$. We consider the angles from the a-axes in the unit circle, thus $A(\theta) = \frac{1}{2}\theta$. Therefore, points can be parameterized by the sector areas, and addition of angles is the same as addition of these areas.

• For group law with parallels see Figure $2(a)$. The following axioms of Abelian (commutative) group hold:

1. Closure: Mapping two points on the circle to another point on the circle.

2. Identity element: I . $P+I=I+P=P$, since the parallel to the line PI that passes through I is the same line, which intersect the circle at P.

3. Inverse element: The inverse of a point P , $(-P)$, is its reflection with the horizontal axes through I. The parallel to the line $P(-P)$ thought I is the tangent to the circle at I, thus its "second" intersection with the circle is I.

4. Associativity: *Pascal's theorem* states that if a hexagon is inscribed in a circle, the three pairs of opposite sides meet at three points which lie on a straight line. A special case of Pascal's theorem: If two pairs of opposite sides of the hexagon inscribed in a circle are parallel then all three pairs of opposite sides are parallel. It follows that $(P+Q)+T = P+(Q+T)$, see Figure $2(b)$. Pascal's theorem is actually even more general - it applies to any conic!

5. Commutativity: The line though P and Q is the same as the line though Q and P, thus $P+Q=Q+P$.

• A quadrilateral can be inscribed in a circle if its opposite angles sum up to π. A trapezoid has two pairs of adjacent angles that sum to π, thus if it is

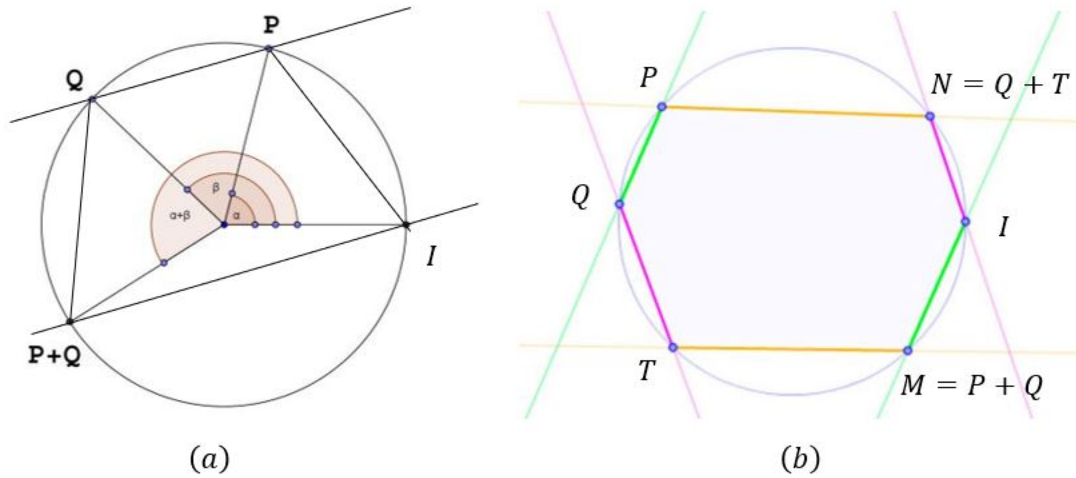

Figure 2: (a) Group law with parallels. (b) By Pascal's theorem $PN \parallel TM$

inscribed in a circle it must be an isosceles trapezoid. Therefore, the chords PI and $Q\,(P+Q)$ in Figure 2(a) are equal and the central angles sum as depicted.

The point is that this construction, that is not defined by angles, is more general, it applies to any conic section. This is because all the group axioms above hold for any conic, but - the kind of group one gets is different for different kinds of conics (ellipse, hyperbola and parabola).

2 Preview to Minkowski Spacetime

In Albert Einstein's original treatment (1905), *the special theory of relativity* is based on two postulates:

1. The laws of physics are invariant (that is, identical) in all inertial frames of reference (that is, frames of reference with no acceleration).

2. The speed of light in vacuum is the same for all observers in different inertial frames.

2.1 The Relativistic Quadrance

First we consider one dimensional space, with coordinate x, and time t. Consider a light beam, in some inertial frame, that travels with constant speed c,

$$\frac{\Delta x}{\Delta t} = c \tag{50}$$

Einstein claims that the speed of light is absolute, not time. The speed of light is invariant under transformation between inertial frames, and time is subordinate to that, it does not have to be invariant. So, let us treat time just as any other coordinate. We manipulate (50) into a quadratic form

$$-\left(c\Delta t\right)^2 + \left(\Delta x\right)^2 = 0 \tag{51}$$

A transformation $(t, x) \to (t', x')$ that preserves (51), preserves the speed of light. We insist on this frame transformation that preserves the l.h.s to be universal, for all particles, which would have nonzero r.h.s.

The relativistic quadrance between two points in *spacetime* plane, $P_1 = (t_1, x_1)$ and $P_2 = (t_2, x_2)$, is defined as

Minkowski quadrance

$$q\left(P_1 P_2\right) = -c^2\left(t_1 - t_2\right)^2 + \left(x_1 - x_2\right)^2 \tag{52}$$

Quadrance (52) defines a non-Euclidean geometry for spacetime, called the *Minkowski plane*. The minus sign makes all the difference, like a seemingly different Pythagoras theorem (is it really a different theorem?). **The group of transformations between inertial frames is the isometry group of Minkowski spacetime**. It is called the *Poincare group*. These are the transformations that preserve the Minkowski quadrance. The **linear** isometries are called ***Lorentz transformations***. They preserve the quadrance of any vector

$$q\left(\boldsymbol{v}\right) = \eta\left(\boldsymbol{v}, \boldsymbol{v}\right) = \boldsymbol{v}^T \eta \boldsymbol{v} = \begin{pmatrix} t & x \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -t^2 + x^2 \tag{53}$$

where the Minkowski (pseudo) inner product is defined by the matrix

$$\eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{54}$$

Then, the Lorentz matrices Λ satisfy

Defining property of Lorentz transformations

$$\Lambda^T \eta \Lambda = \eta \tag{55}$$

and the Lorentz group is the orthogonal group $O\left(1, 1\right)$. You will derive and investigate them in the homework exercise.