HW 2

1 Poiseuille Flow

Poiseuille flow is pressure-induced flow (Channel Flow) in a long duct, usually a pipe.

It is assumed that there is Laminar Flow (flow with no turbulence) of an incompressible Newtonian Fluid of mass density ρ, and viscosity η induced by a constant positive pressure difference or pressure drop Δp .

The fluid flows in a pipe of radius R.

The units of the given parameters are: $[\rho] = \frac{Mass}{length^3}$, $[\eta] = \frac{Mass}{Length\,Time}$, $[\Delta p] = \frac{Mass}{Length^2 Time^2}$.

1. Explain the physical meaning of the given parameters based on their units.
2. Find the dependence of mass flow rate Q in the given paramters.
   Where $[Q] = \frac{Mass}{Time}$ and the dependence on the viscocity is given by $Q \propto \eta^{-1}$.
3. What is the system parameter with the strongest influence on Q?

***Viscosity** is a flow resistance to deformation at a given rate under shear stress. It is sometimes corresponds to the concept of “thickness” and it describes the "internal friction" between the fluid particles.

Solution:

1. R - A unit of length that characterizes the shape of the pipe.
   ρ - The total mass of the fluid particles that are within a certain volume.
   Δp - The pressure difference along the pipe length.
   Therefore units of pressure are: $[p] = \frac{Mass}{Length\,Time^2}$, and pressure is the force that causes a particle to pass trough some cross-sectional area.
2. We need the mass flow rate as a function of the rest of the parameters, let’s write:

$$Q = \frac{R^\alpha \rho^\beta \Delta p^\gamma}{\eta}$$

   Using dimensional analysis:

$$[Q] = \left[\frac{1}{\eta}\right]\left[R^\alpha \rho^\beta \Delta p^\gamma\right] \Rightarrow \frac{M}{T} = \frac{L\,T}{M}\frac{L^\alpha M^{\beta+\gamma}}{L^{3\beta+2\gamma}T^{2\gamma}}$$

we get three equations:

$$\begin{cases} 1+\alpha-(3\beta+2\gamma)=0 \\ \beta+\gamma-1=1 \\ 1-2\gamma=-1 \end{cases}$$

Solve and get: $\alpha = 4$, $\beta = 1$, $\gamma = 1$. So

$$Q \propto \frac{R^4 \rho \Delta p}{\eta}.$$

3. The most contributing factor is the radius of the pipe.

2 Submarine Sinks

A submarine in rest starts to sink according to:

$$\boldsymbol{r}(t) = \frac{\rho}{k}\left(\frac{m}{k}\left[\hat{\boldsymbol{\imath}} + \cos\left(\frac{k}{m}t\right)\hat{\boldsymbol{\jmath}}\right] - \left(t + \frac{m}{k}e^{-\frac{k}{m}t}\right)\hat{\boldsymbol{k}}\right)$$

where ρ, m, and k are constants.

a. Find logical dimensions for the constants ρ, m, and k.

b. Find the velocity of the submarine $\boldsymbol{v}(t)$.

c. Find the velocity at $t = 0$.

d. Show that after a long time $v_z(t)$ approaches to be constant.

Solution:

a. We expect the location vector to have length dimension and the argument of the functions $\cos(x)$ and e^x to be dimensionless. Using dimensional analysis we get:

$$\begin{aligned} \left[\tfrac{\rho m}{k^2}\right] &= Length \\ \left[\tfrac{k}{m}\right] &= (Time)^{-1} \\ \left[\tfrac{\rho}{k}\right] &= \tfrac{Length}{Time} \end{aligned}$$

so logical dimensions will be: $[m] = (Time)^2$, $[k] = Time$, and $[\rho] = Length$.

b. The velocity is defined by $\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt}$:

$$\boldsymbol{v}(t) = \frac{\rho}{k}\left(-\sin\left(\frac{k}{m}t\right)\hat{\boldsymbol{\jmath}} - \left(1 - e^{-\frac{k}{m}t}\right)\hat{\boldsymbol{k}}\right)$$

c. Let us set $t = 0$ for the velocity found in **b**:

$$\boldsymbol{v}(0) = \frac{\rho}{k}\left(-\sin(0)\hat{\boldsymbol{\jmath}} - \left(1 - e^0\right)\hat{\boldsymbol{k}}\right) = \frac{\rho}{k}\left(0\hat{\boldsymbol{\jmath}} - (1-1)\hat{\boldsymbol{k}}\right) = 0$$

as can be expected from a body at rest.

d.

$$\lim_{t\to\infty} v_z(t) = \lim_{t\to\infty} -\frac{\rho}{k}\left(1 - e^{-\frac{k}{m}t}\right) = -\frac{\rho}{k}$$

3 Car Motion

A car is traveling in a straight line x.

It's velocity dependent on time according to the expression $v(t) = c_1 t + c_2 t^3$, where c_1 and c_2 are constants.

At t_0 the car was at x_0.

a. Find the car's acceleration $a(t)$ and location $x(t)$.

b. Find the average velocity during the first T seconds of the movement.

c. Find the average acceleration $\overline{a}$ in the period of time between $t_1 < t < t_2$.

d. Find $v(t_1)$.

$$\begin{array}{cc} c_1 = 3\frac{m}{s^2}, & c_2 = 5\frac{m}{s^4} \\ t_0 = 3\,s, & x_0 = 15\,m \\ T = 10\,s, & t_1 = 10\,s, \quad t_2 = 20\,s \end{array}$$

Solution:

a. The acceleration is defined by $a(t) = \frac{dv(t)}{dt}$ so for the car in this problem:

$$a(t) = c_1 + 3c_2 t^2 = 3\frac{m}{s^2} + 15\frac{m}{s^4}t^2$$

Now for the location of the car let us calculate the integral of $v(t)$:

$$x(t) = \int c_1 t + c_2 t^3\, dt = \frac{c_1}{2}t^2 + \frac{c_2}{4}t^4 + C = \frac{3}{2}\frac{m}{s^2}t^2 + \frac{5}{4}\frac{m}{s^4}t^4 + C$$

and the constant C can be found by the initial condition $x(t_0) = x_0$.

$$x_0 = \frac{3}{2}t_0^2 + \frac{5}{4}t_0^4 + C \Rightarrow C = 15 - \frac{27}{2} - \frac{405}{4}\,m = -99.75\,m$$

And

$$x(t) = -99.75\,m + \frac{3}{2}\frac{m}{s^2}t^2 + \frac{5}{4}\frac{m}{s^4}t^4$$

b. The average velocity for the first 10 seconds is given by:

$$\overline{v} = \frac{\Delta x}{\Delta t} = \frac{x(T) - x(0)}{T - 0} = \frac{-99.75 + \frac{3}{2}100 + \frac{5}{4}10000 - (-99.75)}{10} = 1265\frac{m}{s}$$

c.

$$\overline{a} = \frac{\Delta v}{\Delta t} = \frac{v(t_2) - v(t_1)}{t_2 - t_1} = \frac{3 \times 20 + 5 \times 8000 - (3 \times 10 + 5 \times 1000)}{10} = 3503\frac{m}{s^2}$$

d.

$$v(t = 10\,s) = 3 \times 10 + 5 \times 1000 = 5030\frac{m}{s}$$

4 1D Motion

A particle is moving in the positive of x axis so that its velocity is changing according to
$v_x = c\sqrt{x}$,

where $c > 0$. At $t = 0$ the particle is in $x = 0$.

1. What are the units of the constant c?

2. Find the velocity and acceleration as functions of time.

3. Find the average velocity during the time which is required to pass the distance s.

Solution:

1. $[v_x] = \frac{m}{s} \Rightarrow [c\sqrt{x}] = [c]\,(Length)^{\frac{1}{2}} \Rightarrow [c] = \frac{(Length)^{\frac{1}{2}}}{Time}$.

2.

$$\frac{dx}{dt} = c\sqrt{x} \Rightarrow c\int_0^t dt = \int_0^{x(t)} \frac{dx}{\sqrt{x}}$$

.
One can calculate the derivative: $\frac{d\sqrt{x}}{dx} = \frac{1}{2\sqrt{x}}$.
Solving the integrals:

$$ct = 2\sqrt{x(t)}$$
$$x(t) = \frac{1}{4}c^2t^2$$

Taking a derivative with respect to t for the velocity:

$$v_x(t) = \frac{1}{2}c^2t$$

And again for acceleration:

$$a_x(t) = \frac{1}{2}c^2$$

3. First we need to find the time it takes to pass a distance s:

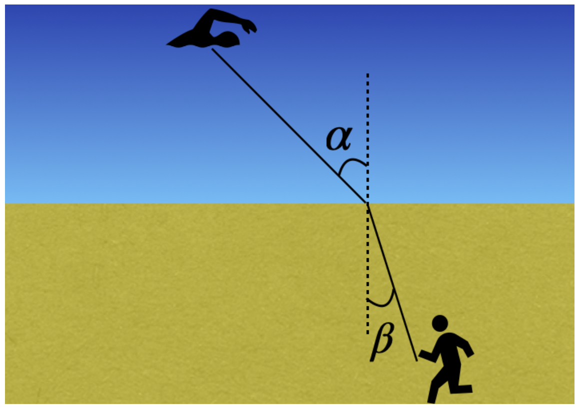
$$s = \frac{1}{4}c^2t_s^2 \Rightarrow t_s = \frac{2\sqrt{s}}{c}\,seconds$$

Then we can calculate the average velocity by:

$$\overline{v} = \frac{s-0}{t_s-0} = \frac{c\sqrt{s}}{2}$$

5 Minimum Time

A safeguard at the beach spots a swimmer in distress and rush to his aid. It is known that the safeguard can run at maximum velocity $v = 5$ m/s and swim at maximum velocity $u = 3$ m/s. What should be the ration between $\sin\alpha$ and $\sin\beta$ (see figure) so that it would take him minimal time to reach the swimmer?

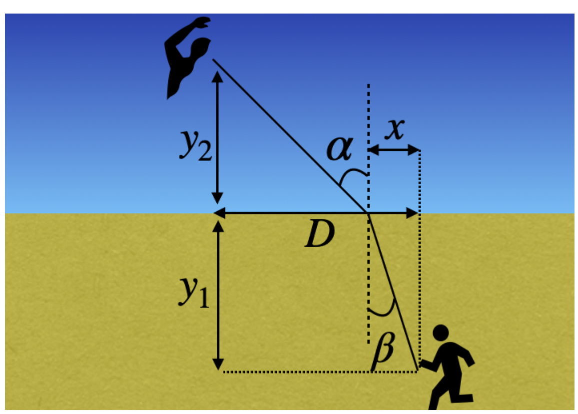


Solution:

We would like to minimize the total time it takes the safeguard to reach the swimmer, therefore let us first write down the expressions for the time it takes running and swimming:

$$\Delta t_1 = \frac{d_1}{v},$$
$$\Delta t_2 = \frac{d_2}{v},$$

where d_1 and d_2 are the distances on the shore and in the water respectively. Since we are looking for the sine functions of the angles, let us parametrize the problem using a horizontal parameter (i.e. parallel to the shore) and define the parameter x as the horizontal distance the safeguard runs. We will denote the total horizontal distance to the swimmer by D and the vertical distances on the shore and in the water as y_1 and y_2 respectively (see figure).



The total time is the sum $T = \Delta t_1 + \Delta t_2$, using simple geometry we can express it as

$$T = \frac{\sqrt{y_1^2 + x^2}}{v} + \frac{\sqrt{y_2^2 + (D-x)^2}}{u},$$

which we minimize with respect to our new parameter x

$$\frac{dT}{dx} = \frac{x}{v\sqrt{y_1^2 + x^2}} - \frac{D-x}{u\sqrt{y_2^2 + (D-x)^2}}$$
$$= \frac{x}{vd_1} - \frac{D-x}{ud_2} = 0.$$

But the ratios between the horizontal parameter and the distance is exactly the sine functions (as we planned), thus

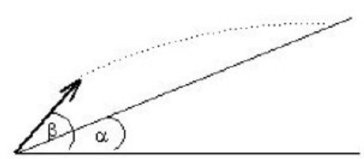
$$\frac{\sin\beta}{v} - \frac{\sin\alpha}{u} = 0 \quad \rightarrow \quad \frac{\sin\alpha}{\sin\beta} = \frac{u}{v}.$$

Do you recognize this relation? This is Snell's law!

6 2D Ballistics - Bonus

A cannonball is fired in an angle β over an inclined plain of angle α.
The ball land 6 meters up the plain, from the bottom.

1. Find the firing velocity of the cannonball v_0.
2. What is the cannonball velocity vector when it hit the plain (express in terms of α, β, and g)?
3. Show that the velocity you found at 2. is directed horizontally if $\tan(\alpha) = \frac{1}{2}\tan(\beta)$



Solution:

1. If we set firing time to be at $t = 0$ the location vector of the cannonball is given by:

$$\boldsymbol{r}(t) = \left(v_0 \cos\beta t, v_0 \sin\beta t - \frac{1}{2}gt^2\right)$$

Now we want to find a value for v_0 for which at some point in time $\boldsymbol{r}(t') = 6\,(\cos\alpha, \sin\alpha)$.

$$\left(v_0 \cos\beta t', v_0 \sin\beta t' - \frac{1}{2}gt'^2\right) = 6\,(\cos\alpha, \sin\alpha)$$

From the x component of the equation:

$$t' = \frac{6\cos\alpha}{v_0\cos\beta}$$

Setting t' into the y component of the equation:

$$6\tan\beta\cos\alpha - \frac{1}{2}g\left(\frac{6\cos\alpha}{v_0\cos\beta}\right)^2 = 6\sin\alpha$$

$$v_0^2 = \frac{3g\cos^2\alpha}{\cos^2\beta\,(\tan\beta\cos\alpha - \sin\alpha)}$$

$$v_0 = \frac{\cos\alpha\sqrt{3g}}{\cos\beta\sqrt{\tan\beta\cos\alpha - \sin\alpha}}$$

2. The velocity at any time t is given by:

$$\boldsymbol{v}(t) = (v_0\cos\beta, v_0\sin\beta - gt)$$

So for t'

$$\boldsymbol{v}(t') = \left(v_0\cos\beta, v_0\sin\beta - \frac{6g\cos\alpha}{v_0\cos\beta}\right)$$

3. Noting that for the velocity to be directed horizontally $v_y(t') = 0$ should apply.

$$v_0\sin\beta - \frac{6g\cos\alpha}{v_0\cos\beta} = 0$$

Multiplying by $v_0\cos\beta$ and setting in v_0 expression, we get:

$$\frac{3g\cos^2\alpha\sin\beta}{\cos\beta\,(\tan\beta\cos\alpha - \sin\alpha)} = 6g\cos\alpha$$

$$\frac{1}{2}\frac{\sin\beta}{\cos\beta} = \frac{1}{\cos\alpha}(\tan\beta\cos\alpha - \sin\alpha)$$

$$\frac{1}{2}\tan\beta = \tan\beta - \tan\alpha$$

$$\tan\alpha = \frac{1}{2}\tan\beta$$

7 Time Dependent Acceleration - Bonus

Particle accelerates from rest according to $a(t) = \frac{2}{3}\left[\text{m/s}^3\right]t$. Given that after 3s the particle reaches distance $g = 27$m from the origin, find the expression for its position.

Solution:

In order to find the position we must integrate over $a(t)$ twice and consider the initial and final conditions: $v(t=0)=0$ and $x(t=3\text{s})=g$. Let us rename the coefficient $A=2/3$ m/s^3, and begin with the velocity

$$v(t)=\int a(t)\,dt=\frac{A}{2}t^2+C,$$

using initial condition for v we find $C=0$ (check dimensions). Integrating again yields

$$x(t)=\int v(t)\,dt=\frac{A}{6}t^3+C,$$

this time we use the final condition for x to find $C=g-\frac{A}{6}(3\text{s})^3$ (check dimensions). Therefore, plugging in all constants

$$\begin{aligned} x(t) &= \frac{1}{9}\left[\text{m/s}^3\right]t^3+27\,[\text{m}]-\frac{1}{9}\left[\text{m/s}^3\right]27\left[\text{s}^3\right] \\ &= \frac{1}{9}\left[\text{m/s}^3\right]t^3+24\,[\text{m}]. \end{aligned}$$