# Gravity 1 - Solution 1

## Euclidean and Spherical Geometries

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## Contents

# 1 Euclidean Geometry

## 1.1 Construction Theorems

1. In the tutorial we constructed an equilateral triangle. We construct two equilateral triangle from both sides of the line segment $AB$, see Figure 1. Since $AC = BC$, $AD = BD$, $CD = CD$, then $\Delta ACD \cong \Delta BCD$. Therefore the angles $\angle C_1 = \angle C_2$. Now since $AC = BC$, $\angle C_1 = \angle C_2$, $CE = CE$, then $\Delta ACE \cong \Delta BCE$. Therefore $AE = BE$, thus $CD$ bisect $AB$. Also, $\angle E_1 = \angle E_2 = \frac{\pi}{2}$.

2. Construct a circle of some radius with $E$ in the center. The circle intersects the line at two points $A, B$ with $AE = BE$. Now construct the bisector of $AB$ as the previous section (again Figure 1), so $CD$ is perpendicular to the given line through $E$.

3. Construct a circle with big enough radius with $F$ in the center. The circle intersects the line at two points $A, B$. Construct the same bisector of $AB$ as before.

4. Construct the perpendicular to the line through the given point, and then construct the perpendicular to the perpendicular thought the same point.

5. See Figure 2. Given a line segment $AB$, draw any other line $\mathcal{L}$ through $A$ and mark 5 successive, equally spaced points $A_1, ...A_5$ along $\mathcal{L}$, using the compass set to any fixed radius. Then connect $A_5$ to $B$ with the straightedge, and draw the parallels to $A_5B$ through $A_4, A_3, A_2, A_1$. By Thales's theorem these parallels divide $AB$ into 5 equal parts, since all the ratios of the segments on $\mathcal{L}$ like $AA_1 : A_1A_2$ are $1 : 1$.

## 1.2 The Parallel Axiom

See Figure 3. Given any triangle with angles $\alpha, \beta, \gamma$, we draw a parallel line $\mathcal{L}$ to some base. From Euclid's fifth axiom alternate angles are equal. $\alpha, \beta, \gamma$ sum to a straight angle, thus the angle sum of the triangle is $\alpha + \beta + \gamma = \pi$.

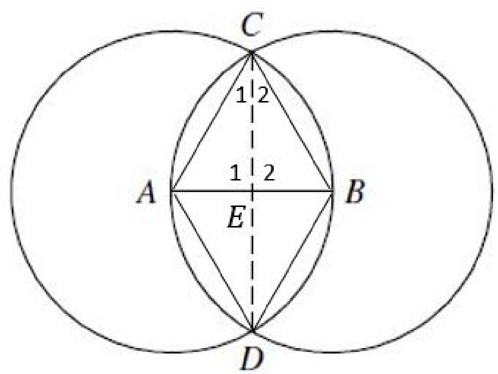


Figure 1: Bisecting a line segment $AB$

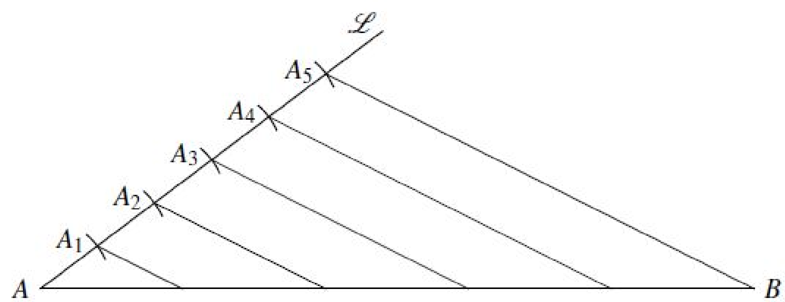


Figure 2: Dividing a line segment into five equal parts

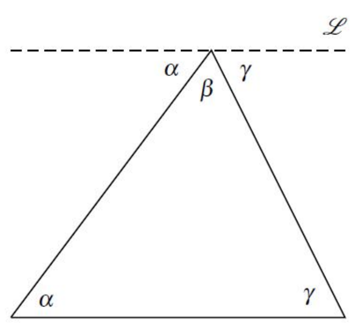


Figure 3: The angle sum of a triangle

# 2 Spherical Geometry

## 2.1 Spherical Law of Cosines

See Figure 4.

The spherical coordinates of point $A$ are $\theta = b$, $\phi = 0$, since the arc length $b$ equals the central angle from the $z$-axes, and $A$ is above the $x$-axes.

$$A = (\sin b, 0, \cos b) \tag{1}$$

The spherical coordinates of point $B$ are $\theta = a$, (same reason) and $\phi = \gamma$. This is because $\gamma$ is the angle between the two planes containing $AC$ and $CB$, which form the same angle down on the $xy$ plane from the $x$ axes to the vertical projection of $B$.

$$B = (\sin a \cos \gamma, \sin a \sin \gamma, \cos a) \tag{2}$$

On the one hand,

$$\boldsymbol{A} \cdot \boldsymbol{B} = |\boldsymbol{A}| \, |\boldsymbol{B}| \cos \angle (\boldsymbol{A}, \boldsymbol{B}) = \cos c \tag{3}$$

and on the other hand

$$\boldsymbol{A} \cdot \boldsymbol{B} = A_x B_x + A_y B_y + A_z B_z = \sin b \sin a \cos \gamma + \cos b \cos a \tag{4}$$

Therefore

$$\boxed{\cos c = \cos a \cos b + \sin a \sin b \cos \gamma} \tag{5}$$

Spherical law of cosines

## 2.2 Harriot's Theorem

### 2.2.1 Arcs of Great Circles

The light rays emanating from the center into an edge of the polyhedron are in a plane through the center. The shadow is the "continuations" of those lines, therefore it is the intersection of this plane and the sphere, which is an arc on a great circle.

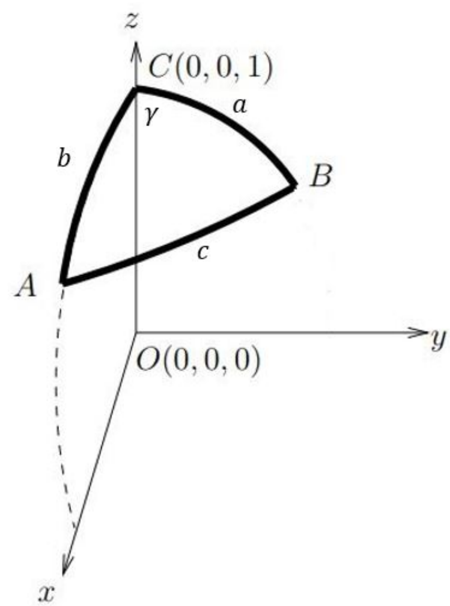


Figure 4: Spherical triangle

#### 2.2.2 Generalized Harriot's Theorem

Consider some quadrilateral $ABCD$ with respective angles $\alpha, \beta, \gamma, \delta$. Connect two opposite vertices $B$ and $D$ with a line, creating two triangles. The angles $\beta$ and $\delta$ are divided into $\beta = \beta_1 + \beta_2$ and $\delta = \delta_1 + \delta_2$, where $\beta_1, \delta_1$ are in one triangle and $\beta_2, \delta_2$ are in the second triangle. We write Harriot's theorem for the two triangles

$$\alpha + \beta_1 + \delta_1 - \pi = \kappa \Delta_{\alpha\beta_1\delta_1} \tag{6}$$

$$\gamma + \beta_2 + \delta_2 - \pi = \kappa \Delta_{\gamma\beta_2\delta_2} \tag{7}$$

and add the two equations

$$\alpha + \gamma + \beta_1 + \beta_2 + \delta_1 + \delta_2 - 2\pi = \kappa \left( \Delta_{\alpha\beta_1\delta_1} + \Delta_{\gamma\beta_2\delta_2} \right) \tag{8}$$

$$\mathcal{E}_{\alpha\beta\gamma\delta} = \kappa A_{\alpha\beta\gamma\delta} \tag{9}$$

For the pentagon, divide it into a quadrilateral and a triangle and sum them.

### 2.2.3 Platonic Tessellations

All the polygons are regular and congruent. We denote the angle of a polygon $\alpha$ and its area $A$. If the number of faces (or edges) meeting at a vertex is $q$, then $\alpha = \frac{2\pi}{q}$. If the number of edges (or vertices) of a polygon is $p$ then the angular excess is $\mathcal{E}_p = p\alpha - (p-2)\pi$. If the total number of faces is $F$ then $A = \frac{S}{F}$ where $S = 4\pi R^2$ is the sphere total area.

- Tetrahedron: $p = 3, q = 3, F = 4$. Therefore $\alpha = \frac{2\pi}{3}$ and $\mathcal{E}_3 = 3\alpha - \pi = \pi$. $A = \frac{S}{4} = \frac{4\pi R^2}{4} = \pi R^2$. $\Rightarrow \mathcal{E} = \frac{1}{R^2}A$.
- Cube: $p = 4, q = 3, F = 6$. Therefore $\alpha = \frac{2\pi}{3}$ and $\mathcal{E}_4 = 4\alpha - 2\pi = \frac{2}{3}\pi$. $A = \frac{S}{6} = \frac{4\pi R^2}{6} = \frac{2}{3}\pi R^2$. $\Rightarrow \mathcal{E} = \frac{1}{R^2}A$.
- Octahedron: $p = 3, q = 4, F = 8$. Therefore $\alpha = \frac{2\pi}{4} = \frac{\pi}{2}$ and $\mathcal{E}_3 = 3\alpha - \pi = \frac{\pi}{2}$. $A = \frac{S}{8} = \frac{4\pi R^2}{8} = \frac{\pi}{2}R^2$. $\Rightarrow \mathcal{E} = \frac{1}{R^2}A$.
- Dodecahedron: $p = 5, q = 3, F = 12$. Therefore $\alpha = \frac{2\pi}{3}$ and $\mathcal{E}_5 = 5\alpha - 3\pi = \frac{\pi}{3}$. $A = \frac{S}{12} = \frac{4\pi R^2}{12} = \frac{\pi}{3}R^2$. $\Rightarrow \mathcal{E} = \frac{1}{R^2}A$.
- Icosahedron: $p = 3, q = 5, F = 20$. Therefore $\alpha = \frac{2\pi}{5}$ and $\mathcal{E}_3 = 3\alpha - \pi = \frac{\pi}{5}$. $A = \frac{S}{20} = \frac{4\pi R^2}{20} = \frac{\pi}{5}R^2$. $\Rightarrow \mathcal{E} = \frac{1}{R^2}A$.

## 2.3 Area of a Circle

### 2.3.1 Area Formula

In Tutorial 1 we derived the circumference of a circle of radius $r$

$$C(r) = 2\pi R \sin\left(\frac{r}{R}\right) \tag{10}$$

To find the area we integrate

$$A(r) = \int_0^r C(r')\,dr' = 2\pi R \int_0^r \sin\left(\frac{r'}{R}\right) dr' = 2\pi R^2 \left[-\cos\left(\frac{r'}{R}\right)\right]_0^r \tag{11}$$

$$\boxed{A(r) = 2\pi R^2 \left(1 - \cos\left(\frac{r}{R}\right)\right)} \tag{12}$$

Area of a spherical circle

### 2.3.2 Analysis

*i.* $r = 0$: $A(0) = 0$, $C(0) = 0$ are the area and circumference of a point.

*ii*. $r = \frac{\pi R}{2}$: $A\left(\frac{\pi R}{2}\right) = 2\pi R^2$, $C\left(\frac{\pi R}{2}\right) = 2\pi R$ are the area of a hemisphere and the circumference of a great circle.

*iii*. $r = \pi R$: $A(\pi R) = 4\pi R^2$, $C(\pi R) = 0$ are the area of the whole sphere and the circumference of a point.

Plot $C(r)$ and $A(r)$.

#### 2.3.3 True or False

*i*. The greater the radius the greater the circumference - false.
*ii*. The greater the radius the greater the area - true.
*iii*. The smaller the circumference the smaller the area - false.
*iv*. A circle with zero circumference has zero radius - false.

#### 2.3.4 Curvature

The non-vanishing zeroth order of (12) in $\frac{r}{R} \ll 1$ is

$$A^{(0)}(r) = 2\pi R^2 \left(1 - \left(1 - \frac{1}{2}\left(\frac{r}{R}\right)^2\right)\right) = \pi r^2 \tag{13}$$

It is the flat Euclidean formula.

The leading order of (12) in $\frac{r}{R} \ll 1$ is the second order

$$A^{(2)}(r) = \pi r^2 - 2\pi R^2 \frac{1}{4!}\left(\frac{r}{R}\right)^4 = \pi r^2 - \frac{\pi}{12}\frac{1}{R^2} r^4 \tag{14}$$

The curvature is

$$\kappa = \frac{1}{R^2} = \frac{12}{\pi}\left(\frac{\pi r^2 - A^{(2)}(r)}{r^4}\right) \tag{15}$$

Curvature as the deviation of the area of a small circle