Gravity 1 - Turorial 4

The Equivalence Principle and Geodesics

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Contents

1 The Equivalence Principle

1.1 Comics

We summarize the equivalence principle in two equivalent pictures. The acceleration is constant and the gravitational field is homogeneous. To go from figure 1 to figure 2, and vise versa, move either the acceleration of the frame or the gravitational field to the other side of the “equation” with a change of sign (direction of the arrow). The red curves are the trajectories of any(!) particle with an initial horizontal velocity, in the given frame. No other forces are present. For inhomogeneous field the equivalence is valid in a small region of space and a short time, i.e., in a small region of spacetime.

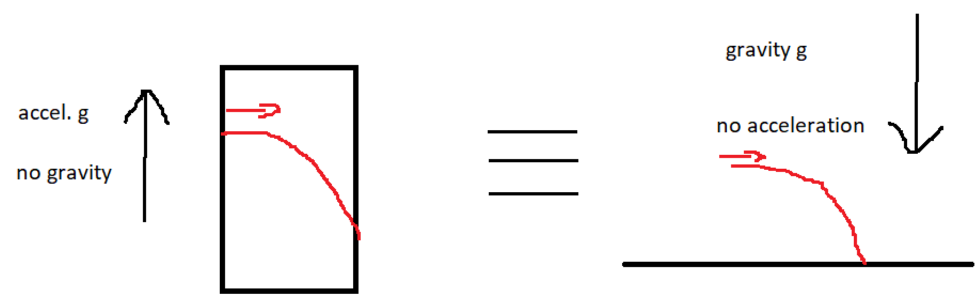

Figure 1: Equivalence principle picture 1

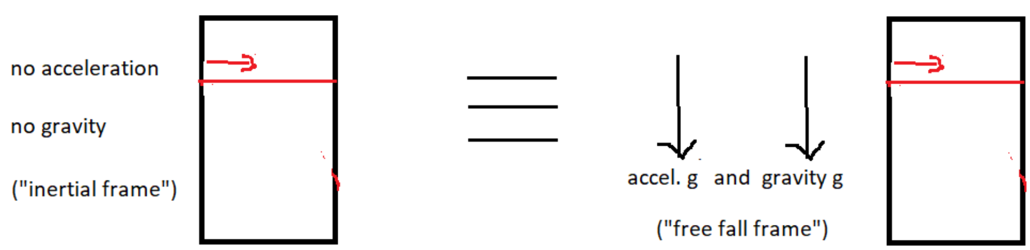

Figure 2: Equivalence principle picture 2

1.2 Uniformly Accelerated Frame

1. Transform the line element of special relativity from the usual Cartesian coordinates (t, x, y, z) to new coordinates (t', x', y', z') related by

$$\begin{aligned} ct &= c\left(\frac{c}{g}+\frac{x'}{c}\right)\sinh\left(\frac{gt'}{c}\right) \\ x &= c\left(\frac{c}{g}+\frac{x'}{c}\right)\cosh\left(\frac{gt'}{c}\right)-\frac{c^2}{g} \\ y &= y' \\ z &= z' \end{aligned} \tag{1}$$

   where g is a constant with the dimensions of acceleration.

2. For $\frac{gt'}{c} \ll 1$ show that this corresponds to a transformation to a uniformly accelerated frame in Newtonian mechanics.

3. Show that an at-rest clock in this frame at $x' = h$ runs fast compared to a clock at rest at $x' = 0$ by a factor $\left(1+\frac{gh}{c^2}\right)$. How is this related to the equivalence principle?

4. Explain the geometric view in spacetime.

1.2.1 The line element in the accelerated frame

The differentials of (1) are

$$\begin{aligned} cdt &= c\left(1+\frac{gx'}{c^2}\right)\cosh\left(\frac{gt'}{c}\right)dt'+\sinh\left(\frac{gt'}{c}\right)dx' \\ dx &= c\left(1+\frac{gx'}{c^2}\right)\sinh\left(\frac{gt'}{c}\right)dt'+\cosh\left(\frac{gt'}{c}\right)dx' \\ dy &= dy' \\ dz &= dz' \end{aligned} \tag{2}$$

The line element is

$$
\begin{aligned}
ds^2 &= -\left(cdt\right)^2 + dx^2 + dy^2 + dz^2 \\
&= -\left(c\left(1+\frac{gx'}{c^2}\right)\cosh\left(\frac{gt'}{c}\right)dt' + \sinh\left(\frac{gt'}{c}\right)dx'\right)^2 \\
&\quad + \left(c\left(1+\frac{gx'}{c^2}\right)\sinh\left(\frac{gt'}{c}\right)dt' + \cosh\left(\frac{gt'}{c}\right)dx'\right)^2 + dy'^2 + dz'^2 \\
&= -c^2\left(1+\frac{gx'}{c^2}\right)^2\cosh^2\left(\frac{gt'}{c}\right)dt'^2 - \sinh^2\left(\frac{gt'}{c}\right)dx'^2 - 2c\left(1+\frac{gx'}{c^2}\right)\sinh\left(\frac{gt'}{c}\right)\cosh\left(\frac{gt'}{c}\right)dt'dx' \\
&\quad +c^2\left(1+\frac{gx'}{c^2}\right)^2\sinh^2\left(\frac{gt'}{c}\right)dt'^2 + \cosh^2\left(\frac{gt'}{c}\right)dx'^2 + 2c\left(1+\frac{gx'}{c^2}\right)\sinh\left(\frac{gt'}{c}\right)\cosh\left(\frac{gt'}{c}\right)dt'dx' \\
&\quad +dy'^2 + dz'^2
\end{aligned}
\tag{3}
$$

$$
\boxed{ds^2 = -\left(1+\frac{gx'}{c^2}\right)^2\left(cdt'\right)^2 + dx'^2 + dy'^2 + dz'^2} \tag{4}
$$

1.2.2 Newtonian approximation

For $\theta \equiv \frac{gt'}{c} \ll 1$, use approximations $\sinh(\theta) \approx \theta$ and $\cosh(\theta) \approx 1 + \frac{1}{2}\theta^2$, approximate (1)

$$
\begin{aligned}
t &\approx \left(\frac{c}{g} + \frac{x'}{c}\right)\left(\frac{gt'}{c}\right) \approx t' \\
x &\approx c\left(\frac{c}{g} + \frac{x'}{c}\right)\left(1 + \frac{1}{2}\left(\frac{gt'}{c}\right)^2\right) - \frac{c^2}{g} \approx x' + \frac{1}{2}gt'^2 \approx x' + \frac{1}{2}gt^2
\end{aligned}
\tag{5}
$$

This is the Newtonian transformation to an accelerated frame with constant acceleration g along the x axis.

1.2.3 Clocks in the accelerated frame

The clocks at rest in the accelerated frame measure the proper time

$$
d\tau^2 = -\frac{1}{c^2}ds^2 \tag{6}
$$

with $dx' = dy' = dz' = 0$. From (4) their proper time is

$$d\tau_{x'=const} = \left(1 + \frac{gx'}{c^2}\right) dt' \tag{7}$$

So in the accelerated frame (t', x', y', z') the clocks at rest at different positions x' measure different time intervals, according to (7). A clock at rest at $x' = 0$ measures

$$d\tau_{x'=0} = dt' \tag{8}$$

while a clock at rest at $x' = h$ measures

$$d\tau_{x'=h} = \left(1 + \frac{gh}{c^2}\right) dt' \tag{9}$$

$$\frac{d\tau_{x'=h}}{d\tau_{x'=0}} = \left(1 + \frac{gh}{c^2}\right) > 1 \tag{10}$$

Thus, the clock higher up in the accelerated frame runs faster. This is an expression of the equivalence principle. The constant accelerated frame along the positive x-axis with acceleration g is equivalent to an inertial frame with a gravitational force along the negative x'-axis, with gravitational potential

$$\phi = gx' \tag{11}$$

. Clocks in a gravitational field at points A and B tick at different rates according to

$$d\tau_B = \left(1 + \frac{\phi_B - \phi_A}{c^2}\right) d\tau_A \tag{12}$$

This is equivalent to (10). Also, plug (11) into (4) and approximate to first order of $\frac{\phi}{c^2}$ to find the metric of weak static field, suited for low velocity particles

$$ds^2 = -\left(1 + \frac{2\phi}{c^2}\right) (cdt')^2 + dx'^2 + dy'^2 + dz'^2 \tag{13}$$

1.2.4 Geometric view

Recall that the circles in Minkowski plane look like hyperbolas. The (t', x') coordinates are (hyperbolic) polar coordinates, where $\left(x' + \frac{c^2}{g}\right)$ is the radius and t' is the (hyperbolic) angle. The worldline of an observer with constant x' is the hyperbola

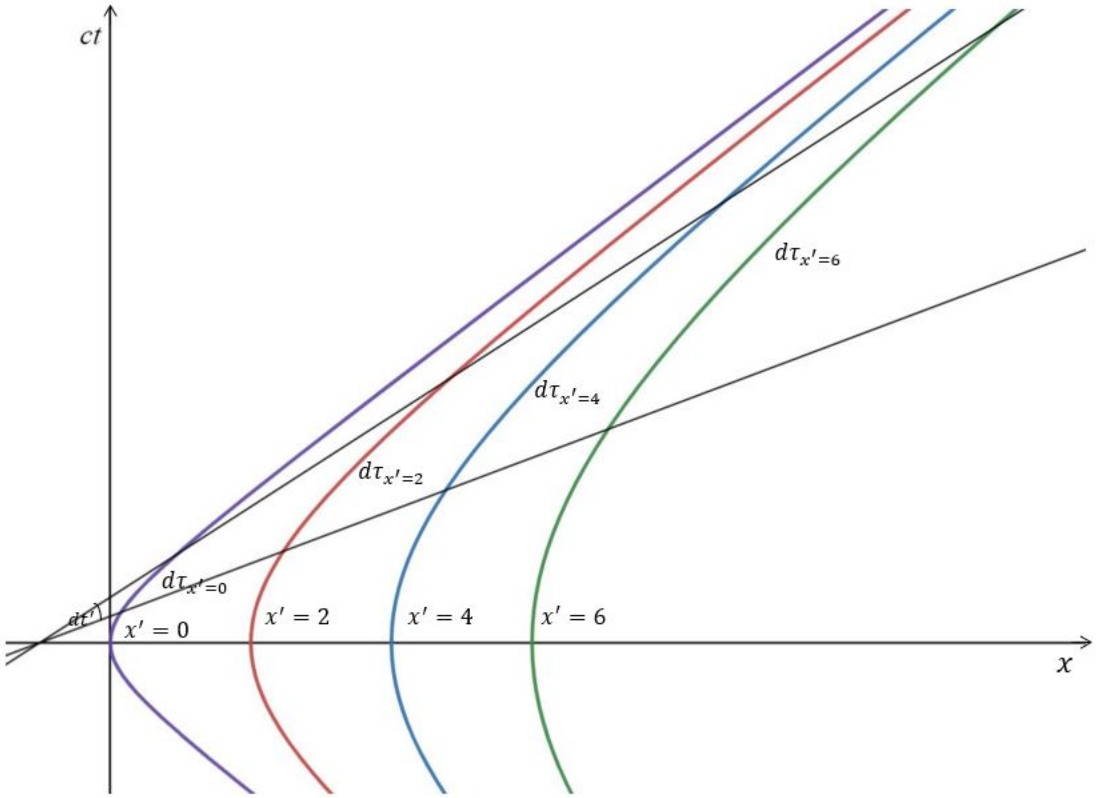

Figure 3: Accelerated frame

$$-\left(ct\right)^2 + \left(x + \frac{c^2}{g}\right)^2 = \left(x' + \frac{c^2}{g}\right)^2 \tag{14}$$

See Figure 3. These are worldlines of constant acceleration g. The coordinates (t', x') are the coordinates used by the observer on $x' = 0$, with proper time $\tau = t'$[1]. The proper time along other constant x' curves are the arc lengths of the same angle. These are just the angle dt' times the radius $\left(1 + \frac{gx'}{c^2}\right)$, see (7). Furthermore, just like for Euclidean circles, the greater the radius the greater the arc length of the same angle.

[1]See also Tutorial 3 section 3 with $x_0 = 0$.

2 Satellite Synchronization

A satellite orbit the earth in a circular motion of radius r. What should that radius be so that the clock on the satellite would be synchronized with a clock on earth (measure the same proper time)? Neglect the earth rotation, and calculate for leading order of weak static field $\frac{\phi}{c^2} \ll 1$ and slow velocity $\frac{\boldsymbol{v}^2}{c^2} \ll 1$. The point is to find the radius of the satellite where the leading effects on its clock from general and special relativity cancel each other.

Solution:

The metric of a weak static field is

$$ds^2 = -\left(1 + \frac{2\phi}{c^2}\right)(cdt)^2 + \left(1 - \frac{2\phi}{c^2}\right)\left(dx^2 + dy^2 + dz^2\right) \tag{15}$$

The proper time squared is

$$\begin{aligned} d\tau^2 &= -\frac{1}{c^2}ds^2 = \left(1 + \frac{2\phi}{c^2}\right)dt^2 - \frac{1}{c^2}\left(1 - \frac{2\phi}{c^2}\right)\left(dx^2 + dy^2 + dz^2\right) \\ &= \left[\left(1 + \frac{2\phi}{c^2}\right) - \frac{1}{c^2}\left(1 - \frac{2\phi}{c^2}\right)\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right)\right]dt^2 \\ &= \left(\left(1 + \frac{2\phi}{c^2}\right) - \frac{1}{c^2}\left(1 - \frac{2\phi}{c^2}\right)\boldsymbol{v}^2\right)dt^2 \\ &\approx \left(\left(1 + \frac{2\phi}{c^2}\right) - \frac{\boldsymbol{v}^2}{c^2}\right)dt^2 \end{aligned} \tag{16}$$

The proper time is

$$d\tau = \left(1 + \frac{2\phi}{c^2} - \frac{\boldsymbol{v}^2}{c^2}\right)^{\frac{1}{2}} dt \approx \left(1 + \frac{\phi}{c^2} - \frac{\boldsymbol{v}^2}{2c^2}\right)dt \tag{17}$$

$$\boxed{d\tau = \left(1 - \frac{1}{c^2}\left(\frac{1}{2}\boldsymbol{v}^2 - \phi\right)\right)dt} \tag{18}$$

The potential on earth is

$$\phi(R) = -\frac{GM}{R} \tag{19}$$

where M is the mass of earth, R is the earth radius and G is Newton's gravitational constant. The observer on earth is at rest $\boldsymbol{v}^2 = 0$ relative to the inertial frame (t, x, y, z) (we neglect the earth rotation and motion). Plug into (18) we find the clock on earth to have

$$d\tau_E = \left(1 - \frac{GM}{c^2 R}\right) dt \tag{20}$$

The clock is modified by the gravity.

The potential where the satellite is

$$\phi\left(r\right) = -\frac{GM}{r} \tag{21}$$

The satellite makes circular motion

$$a_r = \frac{\boldsymbol{v}^2}{r} = \frac{GM}{r^2} \tag{22}$$

Plug into (18) we find the clock on the satellite to have

$$d\tau_S = \left(1 - \frac{1}{c^2}\left(\frac{1}{2}\frac{GM}{r} + \frac{GM}{r}\right)\right) dt = \left(1 - \frac{3}{2}\frac{GM}{c^2 r}\right) dt \tag{23}$$

Synchronize!

$$d\tau_E = d\tau_S \tag{24}$$

$$\left(1 - \frac{GM}{c^2 R}\right) dt = \left(1 - \frac{3}{2}\frac{GM}{c^2 r}\right) dt \tag{25}$$

Therefore

$$\boxed{r = \frac{3}{2} R} \tag{26}$$

3 Geodesics

An action $S\left[q_i\left(t\right), \dot{q}_i\left(t\right)\right]$ is a functional of the coordinates of the particle $q_i\left(t\right)$ and their derivatives (velocities) $\dot{q}_i\left(t\right)$.

$$S = \int dt L \tag{27}$$

The particle moves along a trajectory that extremizes the action $\delta S = 0$. The integrand of the action is called a Lagrangian $L\left(q_i, \dot{q}_i\right)$. Extremizing the action yields the Euler-Lagrange equations for the given Lagrangian

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \tag{28}$$

These are the equations of motion of the particle.

3.1 A Simplified Lagrangian

Free massive particles move between two events in spacetime along worldlines of extremal proper time. These worldlines are geodesics of spacetime, i.e., curves that extremize the proper time/length functional

$$\tau = \int d\tau = \int \sqrt{-ds^2} = \int \sqrt{-g_{\mu\nu}dx^\mu dx^\nu} = \int d\lambda \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}} \tag{29}$$

where $x^\mu(\lambda)$ is a worldline with some parameter λ, and $g_{\mu\nu}$ is the metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{30}$$

Let us denote

$$f \equiv g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \tag{31}$$

The proper time functional thus has the form

$$\tau = \int d\lambda \sqrt{-f} \tag{32}$$

The variation looks like

$$\delta\tau = \int d\lambda \delta\sqrt{-f} = -\frac{1}{2}\int d\lambda f^{-\frac{1}{2}}\delta f \tag{33}$$

It makes things easier if we now specify that our parameter is the proper time τ itself, rather than some arbitrary parameter λ. This fixes the value of f,

$$f = g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = g_{\mu\nu}U^\mu U^\nu = -1 \tag{34}$$

From (33) we have

$$\delta\tau = -\frac{1}{2}\int d\tau \delta f \tag{35}$$

Clearly the following alternative action would have the same variation (up to a minus sign)

$$S = \frac{1}{2}\int d\tau f \tag{36}$$

$$\delta S = \frac{1}{2}\int d\tau \delta f \tag{37}$$

Therefore, a worldline $x^{\mu}(\tau)$ that extremize (36) ($\delta S = 0$) also extremize τ (32) ($\delta\tau = 0$). Plug f (31) into S (36) we have the simple action

$$S = \frac{1}{2}\int d\tau g_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} \tag{38}$$

We can use the Lagrangian

$$\boxed{L = \frac{1}{2}g_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}} \tag{39}$$

Which is the form of a kinetic energy of the particle $\left(\frac{1}{2}U^2\right)$ (per unit mass).

3.2 Classic Examples

Let us find the geodesic equation for some common two-dimensional examples.

Consider the metric of a 2-sphere in spherical coordinates

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2 \tag{40}$$

where θ and ϕ are the polar and azimuthal angles respectively.

Consider also the metric of the Euclidean plane in polar coordinates

$$ds^2 = dr^2 + r^2 d\theta^2 \tag{41}$$

It will be efficient to consider a metric of the form

$$ds^2 = d\chi^2 + f(\chi)^2 d\psi^2 \tag{42}$$

where χ and ψ are some general coordinates, and $f(\chi)$ is some arbitrary function of the the first coordinate only. (42) is a general 2-dimensional diagonal metric which is independent of one of the coordinates.

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & f(\chi)^2 \end{pmatrix} \tag{43}$$

The Lagrangian (39) is

$$L = \frac{1}{2}\left(\left(\frac{d\chi}{d\tau}\right)^2 + f(\chi)^2\left(\frac{d\psi}{d\tau}\right)^2\right) \tag{44}$$

The Euler-Lagrange equations (28) for the metric (42) are

$$\frac{d}{d\tau}\left(\frac{d\chi}{d\tau}\right) - f(\chi) f(\chi)'\left(\frac{d\psi}{d\tau}\right)^2 = 0 \tag{45}$$

$$\frac{d}{d\tau}\left(f(\chi)^2\frac{d\psi}{d\tau}\right) = 0 \tag{46}$$

ψ is a cyclic coordinate, and its conjugate momentum (per unit mass)

$$P_\psi = f(\chi)^2\frac{d\psi}{d\tau} \tag{47}$$

is conserved. Open up (46)

$$f^2\frac{d^2\psi}{d\tau^2} + 2ff'\frac{d\chi}{d\tau}\frac{d\psi}{d\tau} = 0 \tag{48}$$

The equations of motion are

$$\frac{d^2\chi}{d\tau^2} - f(\chi) f(\chi)'\left(\frac{d\psi}{d\tau}\right)^2 = 0 \tag{49}$$

$$\frac{d^2\psi}{d\tau^2} + 2\frac{f(\chi)'}{f(\chi)}\frac{d\chi}{d\tau}\frac{d\psi}{d\tau} = 0 \tag{50}$$

These geodesic equations for the coordinates (χ, ψ) have a “naive” acceleration terms $\frac{d^2\chi}{d\tau^2}, \frac{d^2\psi}{d\tau^2}$ plus additional “corrections”, proportional to the coordinates velocities. The additional term in (49) arise from the explicit dependence of the Lagrangian of the χ coordinate. It is like a “force” term derived from a potential, and it has a minus sign. The additional term in (50) arise from the implicit dependence of the Lagrangian of the χ coordinate, through the motion of the particle in the χ direction. It has a plus sign.

3.2.1 The 2-sphere in spherical coordinates

The metric is

$$ds^2 = d\theta^2 + sin^2\theta d\phi^2 \tag{51}$$

The coordinates are

$$\chi = \theta \qquad \psi = \phi \tag{52}$$

$$f(\theta) = \sin\theta \qquad f(\theta)' = \cos\theta \tag{53}$$

The equations of motion are

$$\frac{d^2\theta}{d\tau^2} - \sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2 = 0 \tag{54}$$

$$\frac{d^2\phi}{d\tau^2} + 2\cot\theta\frac{d\theta}{d\tau}\frac{d\phi}{d\tau} = 0 \tag{55}$$

The solutions are great circles. Any longitudal line $\phi = \phi_0$, $\theta(\tau) = a + b\tau$ is a solution. By rotation symmetry (isometry) we can orient the sphere such that any antipodal points are at the north and south poles, therefore any great circle is a geodesic. Any general circle can be oriented to lie on a latitude with $\theta(\tau) = \theta_0, \phi(\tau) = \phi_0 + b\tau$. This yields the equation $b^2 \sin\theta_0 \cos\theta_0 = 0$. Either $b = 0$, which is the longitude $\phi = \phi_0$, or $\theta_0 = \frac{\pi}{2}$, which is the equator. The motion has constant speed because the choice of the parameter τ. The conserved (angular) momentum (47) is

$$P_\phi = \sin^2\theta\frac{d\phi}{d\tau} \tag{56}$$

which is why the motion is confined in a plane (circle on the sphere).

3.2.2 The Euclidean plane in polar coordinates

The metric is

$$ds^2 = dr^2 + r^2 d\theta^2 \tag{57}$$

The coordinates are

$$\chi = r \qquad \psi = \theta \tag{58}$$

$$f(r) = r \qquad f(r)' = 1 \tag{59}$$

The equations of motion are

$$\frac{d^2 r}{d\tau^2} - r\left(\frac{d\theta}{d\tau}\right)^2 = 0 \tag{60}$$

$$\frac{d^2\theta}{d\tau^2} + \frac{2}{r}\frac{dr}{d\tau}\frac{d\theta}{d\tau} = 0 \tag{61}$$

These equations should be very familiar. Recall that in polar coordinates and orthonormal polar basis, the acceleration vector is

$$\boldsymbol{a} = a_r \hat{r} + a_t \hat{\theta} \tag{62}$$

The radial and tangential accelerations are

$$\begin{aligned} a_r &= \ddot{r} - r\dot{\theta}^2 \\ a_t &= r\ddot{\theta} + 2\dot{r}\dot{\theta} \end{aligned} \tag{63}$$

The geodesic equations (60),(61) are nothing but

$$\begin{aligned} a_r &= 0 \\ \frac{1}{r}a_t &= 0 \end{aligned} \tag{64}$$

i.e., motion with no acceleration

$$\boldsymbol{a} = 0 \tag{65}$$

namely, a uniform motion along a straight line in the plane.

The conserved momentum (47) is

$$P_\theta = r^2 \frac{d\theta}{d\tau} \tag{66}$$