Gravity 1 - Solution 2

Minkowski Plane

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Contents

1 Translations

The relativistic (Minkowski) quadrance between points $P_1 = (t_1, x_1)$ and $P_2 = (t_2, x_2)$ is

$$q(P_1P_2) = -c^2(t_1 - t_2)^2 + (x_1 - x_2)^2 \tag{1}$$

A translation of a point $P = (t, x)$ by a vector (a, b) is

$$T_{a,b}(P) = (t + a, x + b) \tag{2}$$

Show that the quadrance is unchanged under translation

$$\begin{aligned} q(T_{a,b}(P_1)\,T_{a,b}(P_2)) &= -c^2((t_1 + a) - (t_2 + a))^2 + ((x_1 + b) - (x_2 + b))^2 \\ &= -c^2(t_1 - t_2)^2 + (x_1 - x_2)^2 = q(P_1P_2) \end{aligned} \tag{3}$$

2 Lorentz Group $O(1,1)$

2.1 Lorentz Matrices

Lorentz transformations are the linear isometries Λ of Minkowski space. Their defining property is that they preserve the inner product (or quadrance) matrix η

$$\Lambda^T \eta \Lambda = \eta \tag{4}$$

where

$$\eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5}$$

for Minkowski plane. We write a general matrix

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{6}$$

and plug in (4) to find constraints on a, b, c, d.

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} -a & -b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} -a^2 + c^2 & -ab + cd \\ -ab + cd & -b^2 + d^2 \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{7}$$

These are three equations with four unknowns

$$-a^2 + c^2 = -1 \tag{8}$$

$$-b^2 + d^2 = 1 \tag{9}$$

$$-ab + cd = 0 \tag{10}$$

Isolate c from (10)

$$c = \frac{ab}{d} \tag{11}$$

$d \neq 0$, otherwise (9) has no solution. Plug c in (8)

$$-a^2 + \frac{a^2 b^2}{d^2} = -1 \tag{12}$$

multiply by $-d^2$

$$a^2 \left(d^2 - b^2\right) = d^2 \tag{13}$$

and plug in (9)

$$a^2 = d^2 \tag{14}$$

$$a = \pm d \tag{15}$$

Substitute back in (11)

$$c = \pm b \tag{16}$$

(15) and (16) always satisfy (10), while both (8) and (9) become

$$-a^2 + b^2 = -1 \tag{17}$$

The two families of matrices are

Lorentz matrices

$$\Lambda^+ = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \quad \det \Lambda^+ = a^2 - b^2 = 1 \tag{18}$$

$$\Lambda^- = \begin{pmatrix} a & -b \\ b & -a \end{pmatrix} \quad \det \Lambda^- = -a^2 + b^2 = -1 \tag{19}$$

2.1.1 Orthonormal basis

The first column vector should be normalized with a quadrance of −1

$$\eta\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix}\right) = -a^2 + b^2 = -1 \tag{20}$$

The second column vector should be normalized with a quadrance of +1

$$\eta\left(\begin{pmatrix} b \\ a \end{pmatrix}, \begin{pmatrix} b \\ a \end{pmatrix}\right) = -b^2 + a^2 = 1 \tag{21}$$

The first and second column vectors should be orthogonal

$$\eta\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} b \\ a \end{pmatrix}\right) = -ab + ba = 0 \tag{22}$$

The same holds for the row vectors.

2.1.2 Relativistic complex numbers

$$\Lambda^+ = \begin{pmatrix} a & b \\ b & a \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = aI + bK \tag{23}$$

where $K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the “relativistic imaginary number”, with $K^2 = I$. It is an Euclidean reflection about the null line $t = x$.

2.2 Drawing the Group

We have two families of matrices with parameters $a, b \in \mathbb{R}$ with $a^2 - b^2 = 1$. The are two hyperbolas, each one with two branches. Thus, there are a total of 4 non-compact connected components.

We can draw them all on the same coordinate axes by renaming $a \leftrightarrow b$ in matrices Λ^-, $\Lambda^- = \begin{pmatrix} b & -a \\ a & -b \end{pmatrix}$ $\det \Lambda^- = -b^2 + a^2 = -1$.

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \Lambda^+(1,0)$, $T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \Lambda^-(0,-1)$, $P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \Lambda^-(0,1)$, $PT = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \Lambda^+(-1,0)$. See Figure 1.

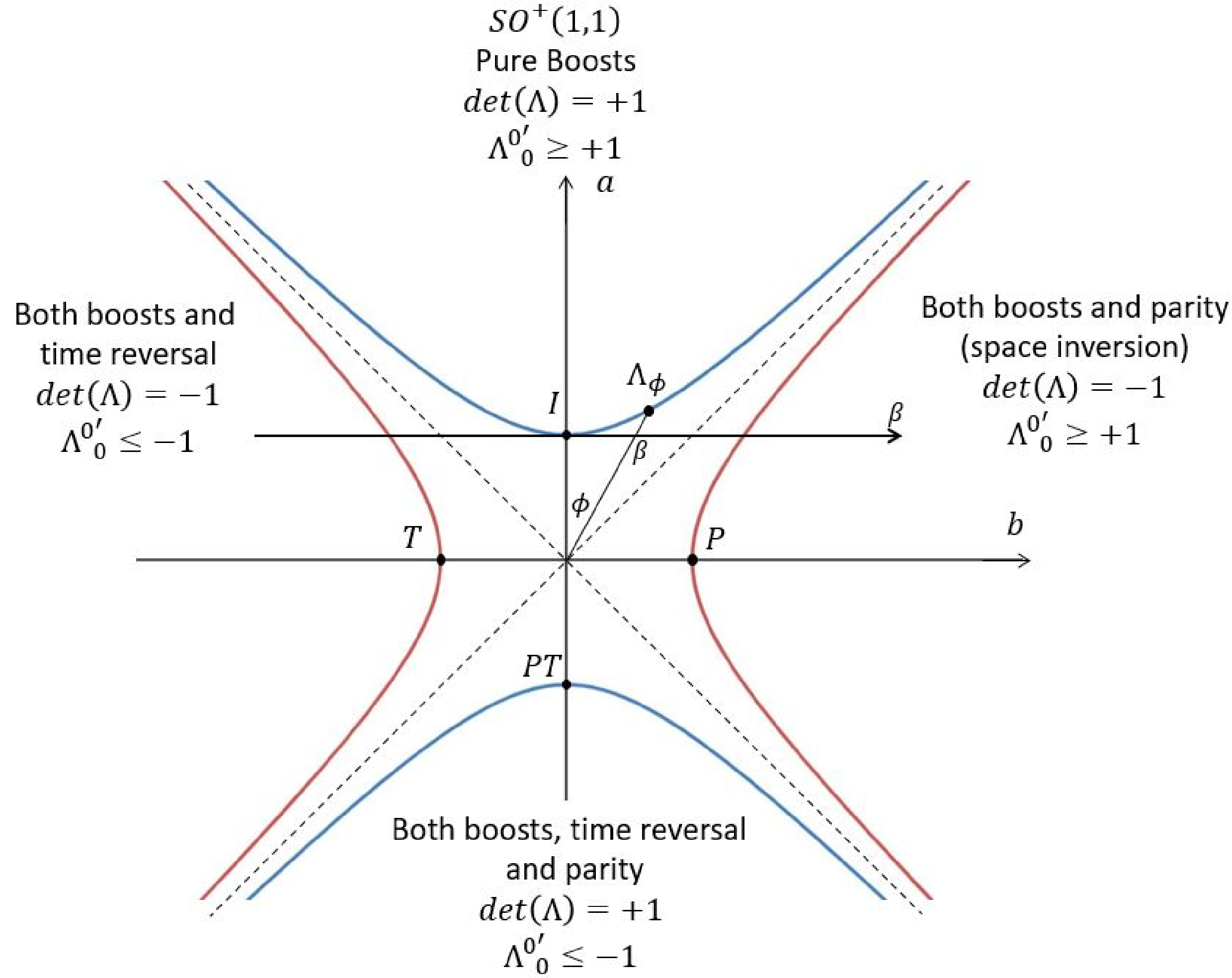

Figure 1: Lorentz group $O\,(1,1)$

2.3 Eigenvectors

We focus on the branch containing the identity, i.e., with $\Lambda = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$, $a^2 - b^2 = 1$ and $a \geq 1$. The characteristic polynomial of Λ is

$$\det(\Lambda - \lambda I) = \begin{vmatrix} a-\lambda & b \\ b & a-\lambda \end{vmatrix} = (a-\lambda)^2 - b^2 = 0 \tag{24}$$

There are two eigen values

$$\lambda_{\pm} = a \pm b \tag{25}$$

If $b = 0$ then $a = 1$ and $\Lambda = \begin{pmatrix} a & b \\ b & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. All the vectors are eigenvectors with $\lambda = 1$. Otherwise, two independent eigenvectors are

$$\boldsymbol{v}_+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad , \quad \boldsymbol{v}_- = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{26}$$

Check

$$\Lambda \boldsymbol{v}_+ = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a+b \\ b+a \end{pmatrix} = (a+b) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \lambda_+ \boldsymbol{v}_+ \tag{27}$$

$$\Lambda \boldsymbol{v}_- = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -a+b \\ -b+a \end{pmatrix} = (a-b) \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \lambda_- \boldsymbol{v}_- \tag{28}$$

Any boost has two independent eigenvectors, as opposed to a general Euclidean rotation which has no eigenvactors. These eigenvectors are "rotating" in the same direction, keeping the same slope. These are the null vectors that generate the light cone. It exhibits the fact that light has the same speed in any inertial frame - its slope in spacetime (velocity) is invariant under Lorentz "rotations".

Lets have another look. If $\boldsymbol{v}$ is an eigenvector of the isometry Λ with eigenvalue λ, then

$$q(\boldsymbol{v}) = q(\Lambda \boldsymbol{v}) = q(\lambda \boldsymbol{v}) = \lambda^2 q(\boldsymbol{v}) \tag{29}$$

$$q(\boldsymbol{v})\left(\lambda^2 - 1\right) = 0 \tag{30}$$

If q is positive-definite ($q(\boldsymbol{v}) > 0$ for any vector and $q(\boldsymbol{v}) = 0$ only for the zero vector), like the Euclidean quadrance, then $\lambda = \pm 1$. Recall the rotations and reflections in the Euclidean plane. But if a nonzero vector can have zero quadrance $q(\boldsymbol{v}) = 0$, then there is no limitation on the eigen value it can have.

2.4 Hyperbolic Parameterization

We parameterize a point on the hyperbola by a hyperbolic angle ϕ (also called rapidity in the relativity literature)

$$a = \cosh\phi \quad , \quad b = \sinh\phi \quad \infty < \phi < \infty \tag{31}$$

The Lorentz boost matrix is

$$\Lambda = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} \tag{32}$$

Hyperbolic rotation (boost) matrix with hyperbolic angle ϕ

Make a rotation by ϕ_1 followed by a rotation by ϕ_2

$$\begin{aligned} \Lambda(\phi_2)\Lambda(\phi_1) &= \begin{pmatrix} \cosh\phi_2 & \sinh\phi_2 \\ \sinh\phi_2 & \cosh\phi_2 \end{pmatrix} \begin{pmatrix} \cosh\phi_1 & \sinh\phi_1 \\ \sinh\phi_1 & \cosh\phi_1 \end{pmatrix} \\ &= \begin{pmatrix} \cosh\phi_2\cosh\phi_1 + \sinh\phi_2\sinh\phi_1 & \cosh\phi_2\sinh\phi_1 + \sinh\phi_2\cosh\phi_1 \\ \sinh\phi_2\cosh\phi_1 + \cosh\phi_2\sinh\phi_1 & \sinh\phi_2\sinh\phi_1 + \cosh\phi_2\cosh\phi_1 \end{pmatrix} \\ &= \begin{pmatrix} \cosh(\phi_1+\phi_2) & \sinh(\phi_1+\phi_2) \\ \sinh(\phi_1+\phi_2) & \cosh(\phi_1+\phi_2) \end{pmatrix} = \Lambda(\phi_1+\phi_2) \end{aligned} \tag{33}$$

2.5 Central Projection Parameterization

2.5.1 Traditional Lorentz Matrix

The center of the hyperbola is at the origin $(0,0)$. We do a central projection of the upper branch of the hyperbola $a^2 - b^2 = 1$ onto the tangent line to the identity $(a = 1, b = 0)$, that is the line $a = 1$. See Figure 1. The parameter on the tangent line is β, which is also the (inverse) slope of the ray $\beta = \frac{b}{a}$. Its range is $-1 < \beta < 1$. We solve the equations of the hyperbola and the ray

$$a^2 - b^2 = 1 \tag{34}$$

$$b = \beta a \tag{35}$$

therefore

$$a^2 - \beta^2 a^2 = a^2\left(1-\beta^2\right) = 1 \tag{36}$$

remember that $a \geq 1$

$$a = \frac{1}{\sqrt{1-\beta^2}} \tag{37}$$

$$b = \frac{\beta}{\sqrt{1-\beta^2}} \tag{38}$$

and the Lorentz matrices are

$$\boxed{\Lambda(\beta) = \begin{pmatrix} \frac{1}{\sqrt{1-\beta^2}} & \frac{\beta}{\sqrt{1-\beta^2}} \\ \frac{\beta}{\sqrt{1-\beta^2}} & \frac{1}{\sqrt{1-\beta^2}} \end{pmatrix}} \tag{39}$$

Lorentz matrices with central projection parameterization

The traditional notation is $\gamma(\beta) = a = \frac{1}{\sqrt{1-\beta^2}}, \gamma \geq 1$,

$$\Lambda(\beta) = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \tag{40}$$

The physical meaning of β, as the slope in the a, b spacetime diagram, is the velocity of the boost $\beta = \frac{v}{c}$. This is physically motivated parameter from a non-relativistic point of view, but for a true relativist it is just some parameter of the isometry group, and not the most convenient one to use (for Euclidean rotations you never wrote the rotation matrices with it).

The coordinate transformation between β and ϕ is

$$\beta = \frac{b}{a} = \frac{\sinh\phi}{\cosh\phi} = \tanh\phi \tag{41}$$

also notice

$$\gamma = \cosh\phi \tag{42}$$

$$\beta\gamma = \sinh\phi \tag{43}$$

2.5.2 Addition of Velocities

$$\begin{aligned}\beta' = \tanh(\phi_1+\phi_2) = \frac{\sinh(\phi_1+\phi_2)}{\cosh(\phi_1+\phi_2)} \\ = \frac{\cosh\phi_2\sinh\phi_1 + \sinh\phi_2\cosh\phi_1}{\cosh\phi_2\cosh\phi_1 + \sinh\phi_2\sinh\phi_1} = \frac{\tanh\phi_1 + \tanh\phi_2}{1+\tanh\phi_1\tanh\phi_2}\end{aligned} \tag{44}$$

$$\beta' = \frac{\beta_1+\beta_2}{1+\beta_1\beta_2} \tag{45}$$

The more famous formula is for the subtraction $\phi_1 - \phi_2$, with $\beta_2 \to -\beta_2$

$$\boxed{\beta' = \frac{\beta_1-\beta_2}{1-\beta_1\beta_2}} \tag{46}$$

(just remember the same sign $\pm$ in the numerator and the denominator, as opposed to the Euclidean case).

2.6 The Hyperbolic Angle and Geometric Group Law

2.6.1 The Geometric Group Law

Consider the hyperbola $xy = 1$, see Figure 2. The identity element on the hyperbola is the vertex $I = (1, 1)$. Given two points on the hyperbola $P = (x_1, y_1)$ and $Q = (x_2, y_2)$, find their "sum" $P + Q = (x_3, y_3)$ obtained by the parallel construction. The slope of the line segment PQ is

$$m_{PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\frac{1}{x_2} - \frac{1}{x_1}}{x_2 - x_1} = \frac{x_1 - x_2}{(x_2 - x_1)\, x_1 x_2} = -\frac{1}{x_1 x_2} \tag{47}$$

The slope of the line segment $I\,(P + Q)$ is

$$m_{I(P+Q)} = \frac{y_3 - 1}{x_3 - 1} = \frac{\frac{1}{x_3} - 1}{x_3 - 1} = \frac{1 - x_3}{(x_3 - 1)\, x_3} = -\frac{1}{x_3} \tag{48}$$

We construct those line segments to be parallel, so their slopes are equal

$$m_{PQ} = m_{I(P+Q)} \tag{49}$$

$$-\frac{1}{x_1 x_2} = -\frac{1}{x_3} \tag{50}$$

Therefore the group law is just multiplication of the coordinates

$$x_3 = x_1 x_2 \tag{51}$$

$$y_3 = y_1 y_2 \tag{52}$$

Algebraically, this is the group $(\mathbb{R} - \{0\}, \times)$, the set of nonzero real numbers with multiplication.

2.6.2 The Hyperbolic Angle

A hyperbolic angle $\Delta\phi$ between two points on a hyperbola is defined as the ratio of the double sector area and the radius R squared $\Delta\phi = \frac{2A}{R^2}$, where R is the Euclidean distance of the vertex from the center. It is the same relation between angle and sector area in a circle. For the hyperbola $xy = 1$, $R^2 = 2$, so

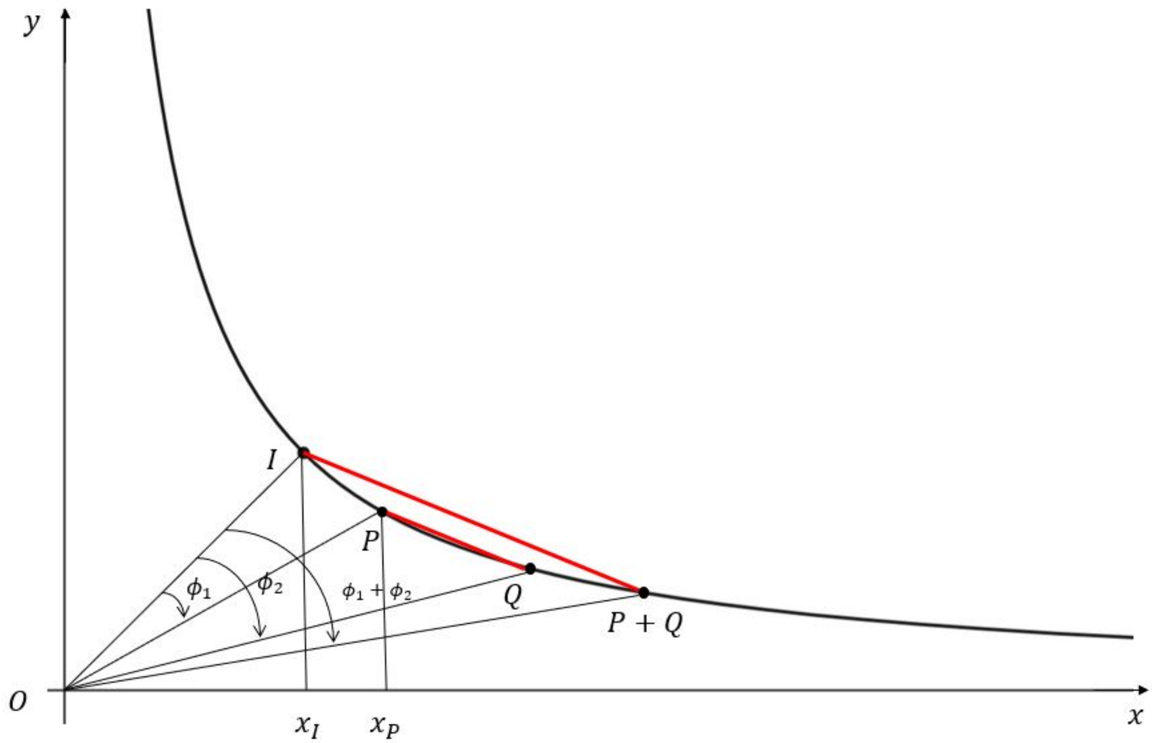

Figure 2: Hyperbola $xy = 1$

the hyperbolic angle between two points is exactly the sector area between them. The angle coordinate ϕ of a point P on the hyperbola is the angle between the vertex of the hyperbola I ($(1, 1)$ in this case) and P, where the area is a signed area (negative to the "left" of I).

Find the hyperbolic angle $\phi(x)$ of a point $P = (x, y)$ on the hyperbola $xy = 1$.

Consider the point P. $x_I = 1$, denote $x_p \equiv x$, $\phi_1 = \phi(x)$. The sector area of $\phi(x)$ is the area under the graph from $x_I = 1$ to $x_P = x$, plus the triangle area $S_{\Delta OIx_I}$ minus the triangle area $S_{\Delta OPx_P}$.

$$\begin{aligned}\phi(x) &= \int_1^x y(x')\,dx' + S_{\Delta OIx_I} - S_{\Delta OPx_P} \\ &= \int_1^x \frac{1}{x'}dx' + \frac{1\cdot 1}{2} - \frac{xy}{2} = \int_1^x \frac{1}{x'}dx' + \frac{1}{2} - \frac{1}{2} \\ &= \ln x - \ln 1 = \ln x \end{aligned} \tag{53}$$

$$\phi(x) = \ln x \tag{54}$$

Thus, the group law for hyperbolic is simple addition

$$\phi_3 = \ln x_3 = \ln x_1 x_2 = \ln x_1 + \ln x_2 = \phi_1 + \phi_2 \tag{55}$$

Therefore the parameterization of x, y with hyperbolic angle is

$$x(\phi) = e^{\phi} \tag{56}$$

$$y(\phi) = e^{-\phi} \tag{57}$$

2.6.3 Coordinate Transformation

In order to transform the hyperbola $xy = 1$ to the form $x'^2 - y'^2 = 1$, we notice that

$$x'^2 - y'^2 = (x' + y')(x' - y') \tag{58}$$

Make the change

$$x = x' + y' \tag{59}$$

$$y = x' - y' \tag{60}$$

then

$$x' = \frac{1}{2}(x + y) \tag{61}$$

$$y' = \frac{1}{2}(x - y) \tag{62}$$

Therefore

$$x'(\phi) = \frac{1}{2}\left(e^{\phi} + e^{-\phi}\right) = \cosh\phi \tag{63}$$

$$y'(\phi) = \frac{1}{2}\left(e^{\phi} - e^{-\phi}\right) = \sinh\phi \tag{64}$$

So we see indeed that the parameter ϕ in our parameterization for the hyperbola of the form $x'^2 - y'^2 = 1$ with hyperbolic functions is the hyperbolic angle as defined by sector area, and that the group $SO^+(1,1)$ (pure boosts/hyperbolic rotations/rotations of Minkowski plane) is the group structure on a hyperbola given by the same geometric construction as for the circle (which yields the rotation group of the Euclidean plane). Remark: this coordinate transformation to coordinates where the hyperbola asymptotes are the axes, is important and useful in relativity. These are called "light-cone" coordinates, since the axes are the light-like directions.

3 Basic Geometry in Minkowski Plane

3.1 Orthogonal Vectors

Consider two vectors $\boldsymbol{v} = \begin{pmatrix} t_1 \\ x_1 \end{pmatrix}$ $\boldsymbol{u} = \begin{pmatrix} t_2 \\ x_2 \end{pmatrix}$. If they are orthogonal then

$$\eta(\boldsymbol{v}, \boldsymbol{u}) = -t_1 t_2 + x_1 x_2 = 0 \tag{65}$$

Their slopes $m_1 = \frac{t_1}{x_1}, m_2 = \frac{t_2}{x_2}$ are related as

$$m_1 m_2 = \frac{t_1 t_2}{x_1 x_2} = \frac{x_1 x_2}{x_1 x_2} = 1 \tag{66}$$

Therefore they are in Euclidean reflection to the $t = \pm x$ axis (disregarding their lengths). Another way to state it is that their angle bisector (in the Euclidean since) is parallel to one of the null lines $t = \pm x$.

The t and x axes are orthogonal, $\eta\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = -0 + 0 = 0$. Null vector is orthogonal to itself. If $q(\boldsymbol{v}) = 0$ then $\eta(\boldsymbol{v}, \boldsymbol{v}) = q(\boldsymbol{v}) = 0$. Thus each light line is orthogonal to itself. However, the two null lines are not orthogonal to each other, $\eta\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right) = -(1)(-1) + (1)(1) = 2 \neq 0$.

3.2 Circles

A circle centered at the origin is the set of all points with constant quadrance

$$q(\boldsymbol{v}) = C \tag{67}$$

In Minkowski plane the circles “look like hyperbolas”

$$-t^2 + x^2 = C \tag{68}$$

For $C > 0$ these are the hyperbolas that cross the $x-axis$. For $C < 0$ these are the hyperbolas that cross the $t-axis$. For $C = 0$ this is a degenerate hyperbola, the two lines $t = \pm x$, the light cone.

Since Lorentz transformation preserves the quadrance, boosting a massive particle moves it along the circle (hyperbola) of negative quadrance centered at the origin it lies on. Boosting a massless particle moves it along the circle (line) of zero quadrance centered at the origin it lies on.

Fact: If a given vector $\boldsymbol{v}$ lies on the circle $q(\boldsymbol{v}) = C$, then any vector $\boldsymbol{u}$ which satisfies the linear equation (in $\boldsymbol{u}$) $g(\boldsymbol{v}, \boldsymbol{u}) = C$ lies on the tangent line to the circle at the point $\boldsymbol{v}$.

Demonstrate it for Euclidean circle $x^2 + y^2 = C$. Let $\boldsymbol{v} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ and $\boldsymbol{u} = \begin{pmatrix} x \\ y \end{pmatrix}$. First we find the tangent line equation at (x_0, y_0) in the standard way using derivative for the slope. Differentiate the circle equation with respect to x

$$2x + 2yy' = 0 \tag{69}$$

At (x_0, y_0)

$$y'(x_0) = -\frac{x_0}{y_0} \tag{70}$$

The tangent line at (x_0, y_0) is given by the equation

$$\begin{aligned} y &= y'(x_0)(x - x_0) + y_0 \\ &= -\frac{x_0}{y_0}(x - x_0) + y_0 \end{aligned} \tag{71}$$

Therefore

$$y_0 y + x_0 x = x_0^2 + y_0^2 \tag{72}$$

which is nothing but

$$\boldsymbol{v} \cdot \boldsymbol{u} = C \tag{73}$$

Now we prove in general (not just for Euclidean case) that the radius vector on a circle is orthogonal to a tangent vector to the circle at that point. Let $\boldsymbol{v}$ be a radius vector on the circle $q(\boldsymbol{v}) = C$. Let $\boldsymbol{u}$ a vector that lies on the

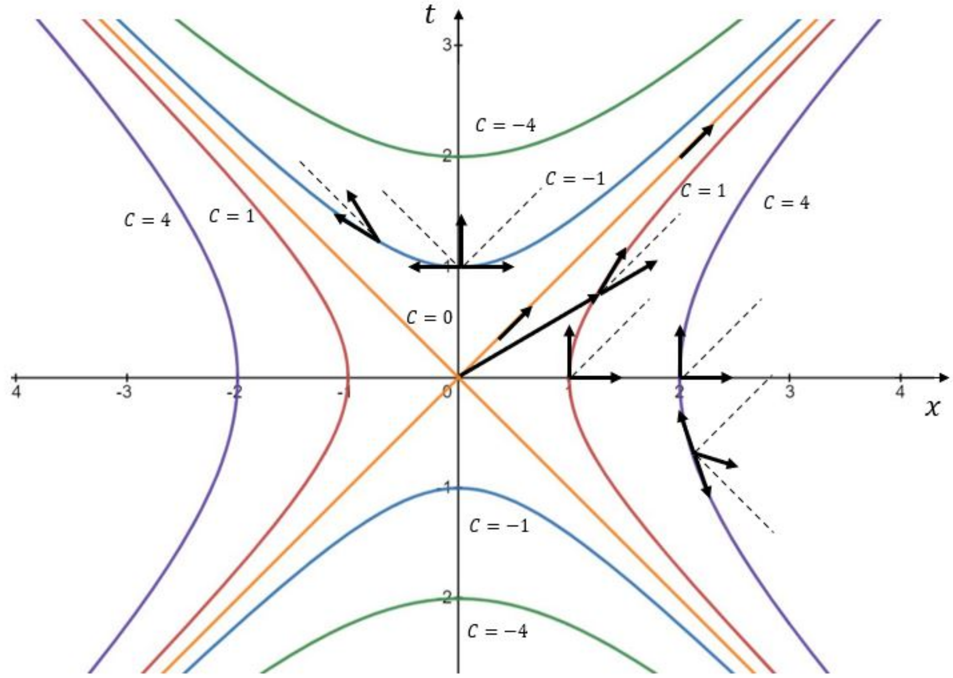

Figure 3: Circles $q(\boldsymbol{v}) = 1,\ q(\boldsymbol{v}) = -1,\ q(\boldsymbol{v}) = 4,\ q(\boldsymbol{v}) = -4,\ q(\boldsymbol{v}) = 0$ with some radial vectors and their orthogonal tangents.

tangent line at $\boldsymbol{v}$. That means that $g(\boldsymbol{v}, \boldsymbol{u}) = C$. Thus the vector $\boldsymbol{u} - \boldsymbol{v}$ is in the direction of the tangent at $\boldsymbol{v}$. Their inner product vanish since

$$g(\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v}) = g(\boldsymbol{v}, \boldsymbol{u}) - g(\boldsymbol{v}, \boldsymbol{v}) = C - C = 0 \tag{74}$$

So the radius is orthogonal to the tangent at the tangency point on the circle.