Gravity 1 - Tutorial 5

The Geodesic Equation

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1 The Geodesic Equation

Free massive particles move between two events in spacetime along worldlines of extremal proper time. These worldlines are geodesics of spacetime, i.e., curves that extremize the proper time/*length functional*

$$\tau = \int d\tau = \int d\lambda \sqrt{-g_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda}} \tag{1}$$

where $x^{\mu}(\lambda)$ is a worldline with some parameter λ, and $g_{\mu\nu}$ is the metric

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} \tag{2}$$

We found that with the choice of the proper time as the parameter of the curve (the arc length parameter), we have the simple action

$$S = \frac{1}{2}\int d\tau g_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} \tag{3}$$

We can use the Lagrangian

$$\boxed{L = \frac{1}{2} g_{\mu\nu}(x)\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}} \tag{4}$$

Which is the form of a kinetic energy of the particle $\left(\frac{1}{2}U^2\right)$ (per unit mass), thus this action is called the *energy functional*.

We want to derive the Euler-Lagrange equations for (4), given a general metric $g_{\mu\nu}(x)$. These equations of motion are called the *geodesic equation*.

Before going through the derivation, let us present some technical facts that will be used during the derivation that follows.

1.1 Some Technical Facts

1. Dummy indices can be renamed. For example

$$T_{\mu\nu} X^{\mu} = T_{\sigma\nu} X^{\sigma} \tag{5}$$

2. Contracting the Kronecker delta with some tensor $T_{\mu\nu}$ just replaces the contracted index

$$T_{\mu\nu}\delta^{\mu}_{\sigma} = T_{\sigma\nu} \tag{6}$$

3. The metric is symmetric. Therefore contracting it with either index is the same, for example

$$g_{\mu\nu}X^{\mu} = g_{\nu\mu}X^{\mu} \tag{7}$$

4. The inverse metric is $g^{\sigma\rho}$ such that

$$g_{\sigma\nu}g^{\sigma\rho} = \delta^{\rho}_{\nu} \tag{8}$$

5. Symmetrization

Let $T_{\mu\nu}$ be some tensor, and $S^{\mu\nu}$ be a symmetric tensor ($S^{\mu\nu} = S^{\nu\mu}$). Take their contraction $T_{\mu\nu}S^{\mu\nu}$. We can relabel the dummy indices $\mu \leftrightarrow \nu$, therefore $T_{\mu\nu}S^{\mu\nu} = T_{\nu\mu}S^{\nu\mu}$. Since S is symmetric, we can exchange back only its indices $T_{\nu\mu}S^{\nu\mu} = T_{\nu\mu}S^{\mu\nu}$. Overall we have

$$T_{\mu\nu}S^{\mu\nu} = T_{\nu\mu}S^{\nu\mu} = T_{\nu\mu}S^{\mu\nu} \tag{9}$$

This result means that a tensor contracted with a symmetric tensor, is itself symmetrized in the given expression.

A second useful trick is to split a tensor with two symmetric indices

$$S^{\mu\nu} = \frac{1}{2}\left(S^{\mu\nu} + S^{\nu\mu}\right) \tag{10}$$

This can be done since if $S^{\mu\nu}$ is symmetric then $\frac{1}{2}\left(S^{\mu\nu} + S^{\nu\mu}\right) = \frac{1}{2}\left(S^{\mu\nu} + S^{\mu\nu}\right) = \frac{1}{2}\left(2S^{\mu\nu}\right) = S^{\mu\nu}$. Therefore (9) can be written as

$$T_{\mu\nu}S^{\mu\nu} = \frac{1}{2}\left(T_{\mu\nu} + T_{\nu\mu}\right)S^{\mu\nu} \tag{11}$$

More general: A tensor $T_{\mu\nu}$ can be decomposed into a symmetric part $T^{(S)}_{\mu\nu} = \frac{1}{2}\left(T_{\mu\nu} + T_{\nu\mu}\right)$ and an anti-symmetric part $T^{(A)}_{\mu\nu} = \frac{1}{2}\left(T_{\mu\nu} - T_{\nu\mu}\right)$ as $T_{\mu\nu} = T^{(S)}_{\mu\nu} + T^{(A)}_{\mu\nu}$. The contractions with a symmetric tensor $S^{\mu\nu}$ and an anti-symmetric tensor $A^{\mu\nu}$ are $T_{\mu\nu}S^{\mu\nu} = T^{(S)}_{\mu\nu}S^{\mu\nu}$ and $T_{\mu\nu}A^{\mu\nu} = T^{(A)}_{\mu\nu}A^{\mu\nu}$. The contraction of a symmetric tensor with an anti-symmetric tensor is zero, proof: $S_{\mu\nu}A^{\mu\nu} = S_{\nu\mu}A^{\nu\mu} = \left(+S_{\mu\nu}\right)\left(-A^{\mu\nu}\right) \Rightarrow 2S_{\mu\nu}A^{\mu\nu} = 0 \Rightarrow S_{\mu\nu}A^{\mu\nu} = 0$.

6. Taking a partial derivative of velocities with respect to velocities of the same coordinate yields $\frac{\partial\left(\frac{dx}{d\tau}\right)}{\partial\left(\frac{dx}{d\tau}\right)} = \frac{\partial\left(\frac{dy}{d\tau}\right)}{\partial\left(\frac{dy}{d\tau}\right)} = ... = 1$. Taking partial derivative of velocities with respect to velocities of some different coordinate yields $\frac{\partial\left(\frac{dx}{d\tau}\right)}{\partial\left(\frac{dy}{d\tau}\right)} =$

$\frac{\partial\left(\frac{dy}{d\tau}\right)}{\partial\left(\frac{dx}{d\tau}\right)} = ... = 0$. In short

$$\frac{\partial\left(\frac{dx^\mu}{d\tau}\right)}{\partial\left(\frac{dx^\nu}{d\tau}\right)} = \delta^\mu_\nu \tag{12}$$

1.2 The Derivation

The Euler-Lagrange equations are

$$\frac{d}{d\tau}\frac{\partial L}{\partial\left(\frac{dx^\sigma}{d\tau}\right)} - \frac{\partial L}{\partial x^\sigma} = 0 \tag{13}$$

The derivative of the Lagrangian (4) with respect to some coordinate x^σ is

$$\frac{\partial L}{\partial x^\sigma} = \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\sigma}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \tag{14}$$

since a general metric tensor is coordinate dependent.

The derivative of the Lagrangian (4) with respect to some velocity $\frac{dx^\sigma}{d\tau}$ is

$$\begin{aligned}\frac{\partial L}{\partial\left(\frac{dx^\sigma}{d\tau}\right)} &= \frac{1}{2}g_{\mu\nu}(x)\frac{\partial}{\partial\left(\frac{dx^\sigma}{d\tau}\right)}\left(\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right) = \frac{1}{2}g_{\mu\nu}\left(\delta^\mu_\sigma\frac{dx^\nu}{d\tau} + \frac{dx^\mu}{d\tau}\delta^\nu_\sigma\right)\\ &= \frac{1}{2}\left(g_{\sigma\nu}\frac{dx^\nu}{d\tau} + g_{\mu\sigma}\frac{dx^\mu}{d\tau}\right) = g_{\sigma\nu}\frac{dx^\nu}{d\tau}\end{aligned} \tag{15}$$

Remark:

Without all the indices, (15) is simply $\frac{\partial}{\partial v}\left(\frac{1}{2}v^2\right) = v$, with one important distinction: The result is the velocity covector, the dual of the velocity vector. Notice that both the l.h.s. and r.h.s of (15) have one free σ index downstairs. This velocity dual vector $g_{\sigma\nu}\frac{dx^\nu}{d\tau}$ is the **canonical momentum** conjugate to x^σ per unit mass[1], while the velocity vector $\frac{dx^\sigma}{d\tau}$ is the **mechanical momentum** per unit mass. If the metric $g_{\mu\nu}(x)$ does not depend on x^σ, then the Lagrangian of a free particle (4) does not depend on x^σ. In this case Euler-Lagrange equation (13) reduces to $\frac{d}{d\tau}\left(\frac{\partial L}{\partial\left(\frac{dx^\sigma}{d\tau}\right)}\right) = \frac{d}{d\tau}\left(g_{\sigma\nu}\frac{dx^\nu}{d\tau}\right) = 0$. Hence, the **dual** velocity along the x^σ direction $U_\sigma = g_{\sigma\nu}\frac{dx^\nu}{d\tau}$ is conserved.

[1] As long as there is no potential term in the Lagrangian.

Moving on, we take a full derivative of (15) with respect to time τ

$$\begin{aligned}\frac{d}{d\tau}\frac{\partial L}{\partial\left(\frac{dx^\sigma}{d\tau}\right)} &= \frac{d}{d\tau}\left(g_{\sigma\nu}\frac{dx^\nu}{d\tau}\right) = g_{\sigma\nu}\frac{d^2x^\nu}{d\tau^2} + \frac{d}{d\tau}\left(g_{\sigma\nu}\left(x^\mu\left(\tau\right)\right)\right)\frac{dx^\nu}{d\tau} \\ &= g_{\sigma\nu}\frac{d^2x^\nu}{d\tau^2} + \frac{\partial g_{\sigma\nu}}{\partial x^\mu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = g_{\sigma\nu}\frac{d^2x^\nu}{d\tau^2} + \frac{1}{2}\left(\frac{\partial g_{\sigma\nu}}{\partial x^\mu} + \frac{\partial g_{\sigma\mu}}{\partial x^\nu}\right)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\end{aligned} \tag{16}$$

where we did the following steps: We took the derivative of the product $\frac{d}{d\tau}\left(g_{\sigma\nu}\frac{dx^\nu}{d\tau}\right)$; Then used the chain rule for $\frac{d}{d\tau}\left(g_{\sigma\nu}\left(x^\mu\left(\tau\right)\right)\right)$; At last we symmetrized $\frac{\partial g_{\sigma\nu}}{\partial x^\mu}$ with respect to μ and ν, since $\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$ is symmetric.

Collecting the two terms (14) and (16) into (13) we have

$$g_{\sigma\nu}\frac{d^2x^\nu}{d\tau^2} + \frac{1}{2}\left(\frac{\partial g_{\sigma\nu}}{\partial x^\mu} + \frac{\partial g_{\sigma\mu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{17}$$

The last step is to get rid of the $g_{\sigma\nu}$ factor in front of the first term. We multiply the equation by $g^{\rho\sigma}$ (raise the index)

$$\frac{d^2x^\rho}{d\tau^2} + \frac{1}{2}g^{\rho\sigma}\left(\frac{\partial g_{\sigma\nu}}{\partial x^\mu} + \frac{\partial g_{\sigma\mu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{18}$$

The geodesic equation

where we used $g^{\rho\sigma}g_{\sigma\nu}\frac{d^2x^\nu}{d\tau^2} = \delta^\rho_\nu\frac{d^2x^\nu}{d\tau^2} = \frac{d^2x^\rho}{d\tau^2}$.

(18) can be written more compactly as

$$\frac{d^2x^\rho}{d\tau^2} + \Gamma^\rho_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{19}$$

where $\Gamma^\rho_{\mu\nu}$ are called the *Christoffel symbols*, and they read

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right) \tag{20}$$

The Christoffel symbols

We used the shorthand notation $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$.

In order to find the geodesic equation of some metric $g_{\mu\nu}$, we can calculate the "Christoffels" directly from the metric (20) and plug them into (19). Notice that $\Gamma^\rho_{\mu\nu}$ is symmetric in its two lower indices. Also, it is build out of first derivatives of the metric. The other direction can be useful too - if we wish to find the Christoffel symbols of some metric, we can derive the geodesic equation by the variational principle and read off the Christoffels from the equation.

2 Christoffel Symbols of a Diagonal Metric

Exercise:

Show that for a **diagonal** metric $g_{\mu\nu}$ the Christoffel symbols are

$$\Gamma^{\rho}_{\mu\nu} = 0 \tag{21}$$

$$\Gamma^{\rho}_{\nu\nu} = -\frac{1}{2} g^{-1}_{\rho\rho} \partial_{\rho} g_{\nu\nu} \tag{22}$$

$$\Gamma^{\rho}_{\nu\rho} = \Gamma^{\rho}_{\rho\nu} = \frac{1}{2} g^{-1}_{\rho\rho} \partial_{\nu} g_{\rho\rho} = \partial_{\nu} \left(ln \sqrt{g_{\rho\rho}} \right) \tag{23}$$

$$\Gamma^{\rho}_{\rho\rho} = \frac{1}{2} g^{-1}_{\rho\rho} \partial_{\rho} g_{\rho\rho} = \partial_{\rho} \left(ln \sqrt{g_{\rho\rho}} \right) \tag{24}$$

where in these expressions $\rho \neq \nu \neq \mu$ and repeated indices are **not** summed over.

3 Classic Examples

Let us find the Christoffel symbols and the geodesic equation for some common two-dimensional examples.

Consider the metric of a 2-sphere in spherical coordinates

$$ds^2 = d\theta^2 + sin^2\theta d\phi^2 \tag{25}$$

where θ and ϕ are the polar and azimuthal angles respectively.

Consider also the metric of the Euclidean plane in polar coordinates

$$ds^2 = dr^2 + r^2 d\theta^2 \tag{26}$$

It will be efficient to consider a metric of the form

$$ds^2 = d\chi^2 + f(\chi)^2 d\psi^2 \tag{27}$$

where χ and ψ are some general coordinates, and $f(\chi)$ is some arbitrary function of the the first coordinate only.

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & f(\chi)^2 \end{pmatrix} \tag{28}$$

The metric components are

$$g_{\chi\chi} = 1 \qquad g_{\psi\psi} = f(\chi)^2 \qquad g_{\chi\psi} = g_{\psi\chi} = 0 \tag{29}$$

All of the first derivatives vanish

$$\partial_\chi g_{\chi\chi} = \partial_\psi g_{\chi\chi} = \partial_\psi g_{\psi\psi} = \partial_\psi g_{\chi\psi} = \partial_\chi g_{\chi\psi} = 0 \tag{30}$$

except for one

$$\partial_\chi g_{\psi\psi} = 2f(\chi)' f(\chi) \tag{31}$$

The only non-vanishing Christoffels are (23),(22)

$$\Gamma^\psi_{\chi\psi} = \Gamma^\psi_{\psi\chi} = \frac{1}{2} g^{-1}_{\psi\psi} \partial_\chi g_{\psi\psi} = \frac{1}{2} f^{-2} (2f'f) = \frac{f(\chi)'}{f(\chi)} \tag{32}$$

$$\Gamma^\chi_{\psi\psi} = -\frac{1}{2} g^{-1}_{\chi\chi} \partial_\chi g_{\psi\psi} = -\frac{1}{2} (2f'f) = -f(\chi)' f(\chi) \tag{33}$$

Summary

$$\boxed{\Gamma^\psi_{\chi\psi} = \Gamma^\psi_{\psi\chi} = \frac{f(\chi)'}{f(\chi)} \qquad \Gamma^\chi_{\psi\psi} = -f(\chi)' f(\chi)} \tag{34}$$

Plug into (19) we get the geodesic equations

$$\frac{d^2\chi}{d\tau^2} - f(\chi) f(\chi)' \left(\frac{d\psi}{d\tau}\right)^2 = 0 \tag{35}$$

$$\frac{d^2\psi}{d\tau^2} + 2\frac{f(\chi)'}{f(\chi)} \frac{d\chi}{d\tau} \frac{d\psi}{d\tau} = 0 \tag{36}$$

3.1 The 2-Sphere in Spherical Coordinates

The metric is

$$ds^2 = d\theta^2 + sin^2\theta d\phi^2 \tag{37}$$

The coordinates are

$$\chi = \theta \qquad \psi = \phi \tag{38}$$

$$f(\theta) = sin\theta \qquad f(\theta)' = cos\theta \tag{39}$$

By (34), the non-vanishing Christoffels are

$$\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \frac{cos\theta}{sin\theta} \qquad \Gamma^{\theta}_{\phi\phi} = -sin\theta cos\theta \tag{40}$$

Plug into (19) we get the geodesic equations

$$\frac{d^2\theta}{d\tau^2} - sin\theta cos\theta \left(\frac{d\phi}{d\tau}\right)^2 = 0 \tag{41}$$

$$\frac{d^2\phi}{d\tau^2} + 2cot\theta \frac{d\theta}{d\tau}\frac{d\phi}{d\tau} = 0 \tag{42}$$

The solutions are great circles.

3.2 The Euclidean Plane in Polar Coordinates

The metric is

$$ds^2 = dr^2 + r^2 d\theta^2 \tag{43}$$

The coordinates are

$$\chi = r \qquad \psi = \theta \tag{44}$$

$$f(r) = r \qquad f(r)' = 1 \tag{45}$$

By (34), the non-vanishing Christoffels are

$$\Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = \frac{1}{r} \qquad \Gamma^{r}_{\theta\theta} = -r \tag{46}$$

Plug into (19) we get the geodesic equations

$$\frac{d^2 r}{d\tau^2} - r\left(\frac{d\theta}{d\tau}\right)^2 = 0 \tag{47}$$

$$\frac{d^2\theta}{d\tau^2} + \frac{2}{r}\frac{dr}{d\tau}\frac{d\theta}{d\tau} = 0 \tag{48}$$

The solutions are straight lines in the plane.