כבידה 1

2022/23 סמסטר א'

פתרון תרגיל 3 – יחסות פרטית

1. טרנספורמצית לורנץ בחלל

(a) According to the formula for a Lorentz transformation

$$\Delta t_{observer} = \gamma \left(\Delta t_{Earth-Sun} - \frac{u}{c^2} \Delta x_{Earth-Sun} \right), \gamma = \frac{1}{\sqrt{1-(\frac{u}{c})^2}}.$$

Plugging in the numbers gives

$$\Delta t_{observer} = \frac{2min - 0.8(8.3min)}{\sqrt{1-0.8^2}} = -7.7min.$$

This means that according to the observer, event B happened before event A! If we reverse the sign of u then

$$\Delta t_{observer} = \frac{2min + 0.8(8.3min)}{\sqrt{1-0.8^2}} = 14min.$$

(b) According to an observer on the spacecraft, $\Delta x_{observer} = 0$.So we can write

$$\begin{aligned} \Delta x_{Earth-Sun} &= \gamma(0 + v\Delta t_{observer}) \\ \frac{\Delta x_{Earth-Sun}}{\Delta t_{observer}} &= v\gamma = \frac{v}{\sqrt{1-(\frac{v}{c})^2}} = \frac{c}{\sqrt{(\frac{c}{v})^2 - 1}} \end{aligned}$$

and solving for v gives

$$\begin{aligned} v &= c\left[1 + \left(\frac{\Delta t_{observer}}{\Delta x_{Earth-Sun/c}}\right)^2\right]^{-1/2} \\ &= (3 \times 10^8 m/s)\left[1 + \left(\frac{5min}{8.3min}\right)^2\right]^{-1/2} = 2.6 \times 10^8 m/s. \end{aligned}$$

(c) In the Earth-Sun frame

$$\Delta t_{Earth-Sun} = \frac{\Delta x_{Earth-Sun}}{v} = \frac{8.3 light - minutes}{2.6 \times 10^8 m/s} = 9.6min.$$

Alternately we can use the time dilation formula to get (the difference is due to rounding errors)

$$\Delta t_{Earth-Sun} = \frac{\Delta x_{Earth-Sun}}{\sqrt{1-(v/c)^2}} = \frac{5min}{\sqrt{1 - \left[1 + (\frac{5min}{8.3min})^2\right]^{-1}}} = 9.7min$$

2. שינויי זמן ומרחק

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \simeq 1 + \frac{1}{2}\beta^2$$

where $\beta = v/c$ and we approximate for small β because the plane is moving much more slowly than light.

Take Δt as the time interval measured by an observer at rest in S. Then

$$\Delta t = \gamma \Delta t'$$

(just plug in a Lorentz transformation with $x = 0$ for rest frame). Then

$$\delta t = \Delta t - \Delta t' \simeq \frac{\beta^2}{2}\Delta t'$$

and for $v = 400\frac{m}{s}$ and $\Delta t' = 3600s$ the difference $\delta t = 3.2ns$.

(b) We assume the meter stick is at rest in S'. As observed by stationary observers in S, the stick moves in the positive x direction with speed v. $x' = \gamma(x - vt)$ relates the position x' measured in S' with the position x measured in X.

Let $\Delta x'$ be the length of the stick measured by an observer at rest in S' (the "proper length"). The stick is moving with speed v in S. To find its length in S, the positions of the front and back of the stick are observed at the same time by two stationary observers in S. The length they measure is the distance Δx between them at $\Delta t = 0$.Then

$$\Delta x' = \gamma \Delta x$$

$$\beta = \sqrt{1 - \left(\frac{\Delta x}{\Delta x'}\right)^2}$$

and for $\Delta x = \Delta x'/2$ you get $\beta = 0.866$. The length of the stick as measured by observers at rest in S is smaller than the length measured by an observer at rest with respect to the stick. ("Lorentz contraction..")

3. בריחה של צופה מכוכב

Solving the EOM for an observer at infinity :

$$\dot{r} = \sqrt{\frac{2GM}{r}} \Rightarrow t = \frac{1}{\sqrt{2GM}}\int_R^r dr\sqrt{r} = \frac{1}{3}\sqrt{\frac{2}{GM}}\left(r^{3/2} - R^{3/2}\right)$$

In observer A frame we should put notice to its velocity (and the velocity change)

$$dt' = dt\sqrt{1-v^2} \Rightarrow t' = \frac{1}{\sqrt{2GM}}\int_R^r dr\sqrt{r\left(1-\frac{2GM}{r}\right)}$$

$$= \frac{1}{\sqrt{2GM}}\int_R^r dr\sqrt{r-2GM}$$

$$= \frac{1}{3}\sqrt{\frac{2}{GM}}\left((r-2GM)^{3/2} - (R-2GM)^{3/2}\right)$$

$$\approx \frac{1}{3}\sqrt{\frac{2}{GM}}\left(r^{3/2} - 3GMr^{1/2} - R^{3/2} + 3GMR^{1/2}\right)$$

$$\Delta t = t - t' \approx \frac{1}{3}\sqrt{\frac{2}{GM}}\left(3GMr^{1/2} - 3GMR^{1/2}\right)$$

$$= \sqrt{2GM}\left(\sqrt{r} - \sqrt{R}\right) \approx \sqrt{2GMr}$$

B. putting c back, we get:

$$\Delta t \approx \frac{1}{c^2}\sqrt{2GMr} \approx 100\mu\sec$$

4. Varying velocity

a) $(gt)^2 < 1 + (gt)^2$ so $dx/dt < 1$.

b)

$$\begin{aligned} u^t &= \frac{1}{\sqrt{1-V^2}} = \sqrt{1+(gt)^2} \\ u^x &= \frac{V}{\sqrt{1-V^2}} = gt \\ u^y &= u^z = 0 \end{aligned}$$

c) The clock of an observer riding on the particle reads proper time. The proper time elapsed from $t = 0$ to t is

$$\tau = \int_0^t dt\,\sqrt{1-V^2} = \int_0^t \frac{dt}{\sqrt{1+(gt)^2}} = \frac{1}{g}\,\sinh^{-1}(gt)\ . \qquad (1)$$

The particle trajectory is

$$x(t) - x_0 = \int_0^t dt\,\frac{gt}{\sqrt{1+(gt)^2}} = \frac{1}{g}\,\sqrt{1+(gt)^2}\ .$$

Thus the relation between τ — the time on the observer's clock — and the location x is

$$\begin{aligned} (x - x_0) &= \frac{1}{g}\,\sqrt{1+\sinh^2(g\tau)} \\ &= \frac{1}{g}\,\cosh(g\tau) \qquad (2) \end{aligned}$$

d) The four force is $f^\alpha = md^2x^\alpha/d\tau^2$ or

$$f^\alpha = (mg\sinh(g\tau), mg\cosh(g\tau), 0, 0)\ .$$

The three force is given by $\vec{F} = md\vec{u}/dt$

$$F^i = (mg, 0, 0)\ .$$

$$f^\alpha = (\gamma F v, \gamma F)$$

$$F = mg$$

Therefore

$$\gamma = cosh\,(g\tau)$$

$$\gamma v = sinh\,(g\tau)$$

$$v = tanh\,(g\tau)$$

Geometric meaning: $\phi = g\tau$ is the hyperbolic angle along the hyperbolic trajectory of the particle in spacetime. It is the instantanuous boost of the particle, relative to an inertial observer. g is the angular frequency.

Kinematic meaning: g is the constant acceleration of the particle.

Also, $1/g$ is the radius of the (hyperbolic) circle in spacetime.

5. Pion Decay

The total 4-momentum before the decay is the 4-momentum of the pion at rest

$$p_i^\alpha = p_\pi^\alpha = (m_\pi, \mathbf{0}) \tag{1}$$

The 4-momentum of the muon is

$$p_\mu^\alpha = (E_\mu, \boldsymbol{P}_\mu) \tag{2}$$

The neutrino is massless so $E_\nu = |\boldsymbol{P}_\nu|$. Its 4-momentum is

$$p_\nu^\alpha = (|\boldsymbol{P}_\nu|, \boldsymbol{P}_\nu) \tag{3}$$

The total 4-momentum after the decay is (2)+(3)

$$p_f^\alpha = p_\mu^\alpha + p_\nu^\alpha = (E_\mu + |\boldsymbol{P}_\nu|, \boldsymbol{P}_\mu + \boldsymbol{P}_\nu) \tag{4}$$

One way to solve is by direct energy-momentum conservation.

4-momentum conservation $p_i^\alpha = p_f^\alpha$

$$(m_\pi, \mathbf{0}) = (E_\mu + |\boldsymbol{P}_\nu|, \boldsymbol{P}_\mu + \boldsymbol{P}_\nu) \tag{5}$$

The momentum conservation $p_i^k = p_f^k$ implies $\boldsymbol{P}_\nu = -\boldsymbol{P}_\mu$. The energy conservation $p_i^0 = p_f^0$ then implies

$$m_\pi = E_\mu + |\boldsymbol{P}_\mu| = \sqrt{m_\mu^2 + |\boldsymbol{P}_\mu|^2} + |\boldsymbol{P}_\mu| \tag{6}$$

$\Rightarrow$

$$(m_\pi - |\boldsymbol{P}_\mu|)^2 = m_\mu^2 + |\boldsymbol{P}_\mu|^2 \tag{7}$$

$\Rightarrow$

$$m_\pi^2 - 2m_\pi |\boldsymbol{P}_\mu| = m_\mu^2 \tag{8}$$

$\Rightarrow$

$$\frac{m_\pi^2 - m_\mu^2}{2m_\pi} = |\boldsymbol{P}_\mu| = m_\mu \gamma |\boldsymbol{v}| = \frac{m_\mu |\boldsymbol{v}|}{\sqrt{1 - \boldsymbol{v}^2}} \tag{9}$$

square the equation, and more algebra yields the solution

$$\boxed{v = \frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2}} \tag{10}$$

A second way to solve is by the conservation of $p^\alpha p_\alpha$ (since p^α is conserved). From (1)

$$p_i^\alpha p_{i\alpha} = -m_\pi^2 \tag{11}$$

from (4) (and momentum conservation $\boldsymbol{P}_\mu + \boldsymbol{P}_\nu = \mathbf{0}$)

$$p_f^\alpha p_{f\alpha} = -(E_\mu + |\boldsymbol{P}_\mu|)^2 = -(\gamma m_\mu + \gamma m_\mu |\boldsymbol{v}|)^2 = -m_\mu^2 \frac{(1 + |\boldsymbol{v}|)^2}{1 - \boldsymbol{v}^2} \tag{12}$$

Since $p_i^\alpha p_{i\alpha} = p_f^\alpha p_{f\alpha}$

$$m_\pi^2 = m_\mu^2 \frac{(1 + |\boldsymbol{v}|)^2}{1 - \boldsymbol{v}^2} \tag{13}$$

Again, algebra yields (10).

6. The Electromagnetic Field

a. Lorentz Transformation

The electromagnetic field tensor $F^{\mu\nu}$, also called the *field strength* tensor of the electromagnetic field, in an inertial frame is

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \tag{14}$$

It is an *antisymmetric* tensor

$$F^{\mu\nu} = -F^{\nu\mu} \tag{15}$$

What is the magnetic field $B^{i'}$ in another inertial frame, moving with relative velocity v along the x axis?

To find out, we transform the tensor $F^{\mu\nu}$ and look at the desired components. A $\binom{2}{0}$-tensor transformation rule is

$$F^{\mu'\nu'} = \Lambda^{\mu'}{}_{\rho}\Lambda^{\nu'}{}_{\sigma}F^{\rho\sigma} \tag{16}$$

where

$$\Lambda^{\mu'}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{17}$$

We work with units $c = 1$, so $\beta = v$. When making index calculation we notice which components vanish, so not to write the whole sum explicitly.

The component parallel to v transforms

$$\begin{aligned} B'_x = F^{2'3'} &= \Lambda^{2'}{}_{\rho}\Lambda^{3'}{}_{\sigma}F^{\rho\sigma} \\ &= \Lambda^{2'}{}_{2}\Lambda^{3'}{}_{3}F^{23} = B_x \end{aligned} \tag{18}$$

The y - component transforms

$$\begin{aligned} B'_y = F^{3'1'} &= \Lambda^{3'}{}_{\rho}\Lambda^{1'}{}_{\sigma}F^{\rho\sigma} \\ &= \Lambda^{3'}{}_{3}\Lambda^{1'}{}_{0}F^{30} + \Lambda^{3'}{}_{3}\Lambda^{1'}{}_{1}F^{31} \\ &= -\gamma v\left(-E_z\right) + \gamma B_y = \gamma\left(B_y - \left(v_z E_x - v_x E_z\right)\right) \\ &= \gamma\left(\boldsymbol{B} - \boldsymbol{v} \times \boldsymbol{E}\right)_y \end{aligned} \tag{19}$$

The z - component transforms

$$\begin{aligned} B'_z = F^{1'2'} &= \Lambda^{1'}{}_{\rho}\Lambda^{2'}{}_{\sigma}F^{\rho\sigma} \\ &= \Lambda^{1'}{}_{0}\Lambda^{2'}{}_{2}F^{02} + \Lambda^{1'}{}_{1}\Lambda^{2'}{}_{2}F^{12} \\ &= -\gamma v E_y + \gamma B_z = \gamma\left(B_z - \left(v_x E_y - v_y E_x\right)\right) \\ &= \gamma\left(\boldsymbol{B} - \boldsymbol{v} \times \boldsymbol{E}\right)_z \end{aligned} \tag{20}$$

where we used the fact that

$$v^i = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix} \tag{21}$$

and

$$(\boldsymbol{v} \times \boldsymbol{E})^i = \begin{pmatrix} v_y E_z - v_z E_y \\ v_z E_x - v_x E_y \\ v_x E_y - v_y E_x \end{pmatrix} \tag{22}$$

We conclude, in general that

Lorentz transformation of the magnetic field

$$\boldsymbol{B}'_{\shortparallel} = \boldsymbol{B}_{\shortparallel} \tag{23}$$

$$\boldsymbol{B}'_{\perp} = \gamma\left(\boldsymbol{B}_{\perp} - \boldsymbol{v} \times \boldsymbol{E}\right) \tag{24}$$

where $\shortparallel$ and $\perp$ mean parallel and perpendicular to $\boldsymbol{v}$. These are 3-vector equations, so they have the same form when rotating the system.

b. Lagrangian Density

$$F_{\mu\nu} = \eta_{\mu\rho}\eta_{\nu\sigma}F^{\rho\sigma} \tag{25}$$

$$F_{0i} = \eta_{0\rho}\eta_{i\sigma}F^{\rho\sigma} = -F^{0i} = -E_i \tag{26}$$

$$F_{ij} = \eta_{i\rho}\eta_{j\sigma}F^{\rho\sigma} = F^{ij} \tag{27}$$

Open the summation, with $i \neq j$,

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \\
&= -\frac{1}{4}\left(F_{0i}F^{0i} + F_{i0}F^{i0} + F_{ij}F^{ij} + F_{ji}F^{ji}\right) \\
&= -\frac{1}{4}\left(F_{0i}F^{0i} + \left(-F_{0i}\right)\left(-F^{0i}\right) + F_{ij}F^{ij} + \left(-F_{ij}\right)\left(-F^{ij}\right)\right) \\
&= -\frac{1}{4}\left(2F_{0i}F^{0i} + 2F_{ij}F^{ij}\right) \\
&= -\frac{1}{4}\left(2\left(-E_i\right)E^i + 2\left(\left(F_{12}\right)^2 + \left(F_{13}\right)^2 + \left(F_{23}\right)^2\right)\right) \\
&= -\frac{1}{4}\left(-2\boldsymbol{E}^2 + 2\boldsymbol{B}^2\right) = \frac{1}{2}\left(\boldsymbol{E}^2 - \boldsymbol{B}^2\right)
\end{aligned} \tag{28}$$

Lorentz transformations of the tensors

$$F^{\mu'\nu'} = \Lambda^{\mu'}_{\ \ \rho}\Lambda^{\nu'}_{\ \ \sigma}F^{\rho\sigma} \tag{29}$$

$$F_{\mu'\nu'} = \Lambda^{\alpha}_{\ \ \mu'}\Lambda^{\beta}_{\ \ \nu'}F_{\alpha\beta} \tag{30}$$

where $\Lambda^{\alpha}_{\ \ \mu'}$ is the inverse matrix of $\Lambda^{\mu'}_{\ \ \rho}$,

$$\Lambda^{\alpha}_{\ \ \mu'}\Lambda^{\mu'}_{\ \ \rho} = \delta^{\alpha}_{\rho} \tag{31}$$

Therefore the full contraction of the tensors is invariant under Lorentz transformation (it is a "Lorentz scalar")

$$\begin{aligned} F_{\mu'\nu'}F^{\mu'\nu'} &= \Lambda^{\alpha}{}_{\mu'}\Lambda^{\beta}{}_{\nu'}\Lambda^{\mu'}{}_{\rho}\Lambda^{\upsilon'}{}_{\sigma}F_{\alpha\beta}F^{\rho\sigma} \\ &=\delta^{\alpha}_{\rho}\delta^{\beta}_{\sigma}F_{\alpha\beta}F^{\rho\sigma} = F_{\rho\sigma}F^{\rho\sigma} = F_{\mu\nu}F^{\mu\nu} \end{aligned} \tag{32}$$

7. Doppler Effect in Galaxy

א. ענן הגז המתקרב אלינו יהיה בעל תדירות גבוהה יותר ולכן אורך הגל הנמוך יותר (העקומה הכחולה).

ב. נרשום את חוק דופלר עבור התדירויות:

$$\left.\begin{aligned} f_+ &= \sqrt{\frac{1+v}{1-v}}f \\ f_- &= \sqrt{\frac{1-v}{1+v}}f \end{aligned}\right\} \Rightarrow f_- f_+ = f^2 \Rightarrow f = \sqrt{f_- f_+} = \frac{c}{\sqrt{\lambda_- \lambda_+}} \approx 5.99 \cdot 10^{14}\, Hz$$

ג. מכיוון שההפרש בין אורכי הגל הוא קטן, ניתן להשתמש בקירוב הלא יחסותי.

$$\lambda_+ - \lambda_- = v(\lambda_+ + \lambda_-) \Rightarrow v = c\frac{\lambda_+ - \lambda_-}{\lambda_+ + \lambda_-} \approx 540\, {}^{km}\!/_{\sec}$$