Gravity 1 - Tutorial 6

The Hyperbolic Plane

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Contents

The ***Hyperbolic plane*** is a non-Euclidean geometry with the first four of Euclid's axioms satisfied, and the *parallel postulate* of Euclidean geometry replaced by:

For any given line L and point P not on L, there are **at least two** distinct lines through P that do not intersect L.

1 The Beltrami-Poincare Half-Plane

The *hyperbolic plane* H^2 can be defined by the metric

$$ds^2 = \frac{R^2}{y^2}\left(dx^2 + dy^2\right) \tag{1}$$

$$y > 0 \tag{2}$$

This is a particular **model** of the hyperbolic plane, called the *Poincare half plane*. It is customary to take the constant length scale R to be 1.

1. Show that points on the x-axis are at infinite distance from any other point (x, y) in the upper half-plane.
2. Calculate the Christoffel symbols directly from the metric, and write down the geodesic equations.
3. Derive the geodesic equations by a variational principle from the Lagrangian, and read off the Christoffel symbols.
4. Use integrals of motion to find the algebraic form of the geodesics, and see that they are semi-circles centered on the x-axis or vertical lines.
5. Find x and y as functions of the length parameter s along the geodesics.

1.1 Infinite Distance

The distance along a curve C of constant x (vertical line with $dx = 0$), towards a point on the x-axis ($y = 0$) is

$$\left|\int_C ds\right| = \left|\int_{y_p}^0 \frac{dy}{y}\right| = \left|\ln(y)\,|_{y_p}^0\right| = \left|\ln(0) - \ln(y_p)\right| \tag{3}$$

which diverges. Similarly, the distance along any other curve to a point on the x-axis will diverge.

1.2 Direct Calculation of the Christoffel Symbols

From (1) we read off he metric g_{ij}

$$\begin{aligned} g_{xx} = g_{yy} &= \frac{1}{y^2} \\ g_{xy} = g_{yx} &= 0 \end{aligned} \tag{4}$$

The inverse metric g^{ij} is

$$\begin{aligned} g^{xx} = g^{yy} &= y^2 \\ g^{xy} = g^{yx} &= 0 \end{aligned} \tag{5}$$

The formula for Christoffel symbols is

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right) \tag{6}$$

Since the metric is diagonal,

$$\Gamma^x_{ij} = \frac{1}{2} g^{xx} \left(\partial_i g_{xj} + \partial_j g_{xi} - \partial_x g_{ij} \right) \tag{7}$$

$$\Gamma^y_{ij} = \frac{1}{2} g^{yy} \left(\partial_i g_{yj} + \partial_j g_{yi} - \partial_y g_{ij} \right) \tag{8}$$

The metric also does not depend on x, so the different components of (7) are

$$\begin{aligned} \Gamma^x_{xx} &= \frac{1}{2} g^{xx} \left(\partial_x g_{xx} + \partial_x g_{xx} - \partial_x g_{xx} \right) = 0 \\ \Gamma^x_{yy} &= \frac{1}{2} g^{xx} \left(\partial_y g_{xy} + \partial_y g_{xy} - \partial_x g_{yy} \right) = 0 \\ \Gamma^x_{xy} &= \frac{1}{2} g^{xx} \left(\partial_x g_{xy} + \partial_y g_{xx} - \partial_x g_{xy} \right) = \frac{1}{2} g^{xx} \partial_y g_{xx} \\ &= \frac{1}{2} y^2 \partial_y y^{-2} = -y^2 y^{-3} = -\frac{1}{y} \end{aligned} \tag{9}$$

and the different components of (8) are

$$\Gamma^y_{xx} = -\frac{1}{2} g^{yy} \partial_y g_{xx} = -\frac{1}{2} y^2 \partial_y y^{-2} = \frac{1}{y} \tag{10}$$

$$\Gamma^y_{yy} = \frac{1}{2} g^{yy} \partial_y g_{yy} = \frac{1}{2} y^2 \partial_y y^{-2} = -\frac{1}{y} \tag{11}$$

$$\Gamma^y_{xy} = \frac{1}{2} g^{yy} \left(\partial_x g_{yy} + \partial_y g_{yx} - \partial_y g_{xy} \right) = 0$$

We calculated all the 6 independent components of Γ^k_{ij}. The non-vanishing Christoffels are

$$\boxed{\Gamma^x_{xy} = \Gamma^x_{yx} = \Gamma^y_{yy} = -\frac{1}{y} \qquad \Gamma^y_{xx} = \frac{1}{y}} \tag{12}$$

Christoffels of the Poincare half-plane

The geodesic equation is

$$\frac{d^2 x^k}{ds^2} + \Gamma^k_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} = 0 \tag{13}$$

Plug in (12) yields the geodesic equations of the Poincare half-plane

$$\frac{d^2 x}{ds^2} - \frac{2}{y} \frac{dx}{ds} \frac{dy}{ds} = 0 \tag{14}$$

$$\frac{d^2 y}{ds^2} + \frac{1}{y} \left(\frac{dx}{ds} \right)^2 - \frac{1}{y} \left(\frac{dy}{ds} \right)^2 = 0 \tag{15}$$

1.3 Direct Calculation of the Geodesic Equation

Let us use the Lagrangian

$$L = \frac{1}{2} g_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} \tag{16}$$

Substitution of the metric components (4) yields

$$L = \frac{1}{2y^2} \left(\frac{dx}{ds} \right)^2 + \frac{1}{2y^2} \left(\frac{dy}{ds} \right)^2 \tag{17}$$

The Euler Lagrange equations are

$$\frac{d}{ds} \frac{\partial L}{\partial \left(\frac{dx^i}{ds} \right)} - \frac{\partial L}{\partial x^i} = 0 \tag{18}$$

We start with the x coordinate

$$\frac{\partial L}{\partial x} = 0 \tag{19}$$

$$\frac{d}{ds}\frac{\partial L}{\partial\left(\frac{dx}{ds}\right)} = \frac{d}{ds}\left(\frac{1}{y^2}\frac{dx}{ds}\right) = 0 \tag{20}$$

Then the x coordinate E-L equation is

$$\frac{1}{y^2}\frac{d^2x}{ds^2} - 2y^{-3}\frac{dx}{ds}\frac{dy}{ds} = 0 \tag{21}$$

Multiplying by y^2

$$\frac{d^2x}{ds^2} - \frac{2}{y}\frac{dx}{ds}\frac{dy}{ds} = 0 \tag{22}$$

Now we turn to the y coordinate

$$\frac{\partial L}{\partial y} = -y^{-3}\left(\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2\right) \tag{23}$$

$$\frac{d}{ds}\frac{\partial L}{\partial\left(\frac{dy}{ds}\right)} = \frac{d}{ds}\left(\frac{1}{y^2}\frac{dy}{ds}\right) = \frac{1}{y^2}\frac{d^2y}{ds^2} - 2y^{-3}\left(\frac{dy}{ds}\right)^2 \tag{24}$$

Then the y coordinate E-L equation is

$$\frac{1}{y^2}\frac{d^2y}{ds^2} - 2y^{-3}\left(\frac{dy}{ds}\right)^2 + y^{-3}\left(\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2\right) = 0 \tag{25}$$

$\Rightarrow$

$$\frac{1}{y^2}\frac{d^2y}{ds^2} + y^{-3}\left(\left(\frac{dx}{ds}\right)^2 - \left(\frac{dy}{ds}\right)^2\right) = 0 \tag{26}$$

Multiplying by y^2

$$\frac{d^2y}{ds^2} + \frac{1}{y}\left(\left(\frac{dx}{ds}\right)^2 - \left(\frac{dy}{ds}\right)^2\right) = 0 \tag{27}$$

We collect results (22),(27)

$$\frac{d^2x}{ds^2} - \frac{2}{y}\frac{dx}{ds}\frac{dy}{ds} = 0 \quad (28)$$

$$\frac{d^2y}{ds^2} + \frac{1}{y}\left(\left(\frac{dx}{ds}\right)^2 - \left(\frac{dy}{ds}\right)^2\right) = 0 \quad (29)$$

The geodesic equations of the Poincare half-plane

Of course we got the same result as in the previous section (14),(15). We read off the Γ^x_{ij} Christoffel symbols from (28) and the Γ^y_{ij} Christoffel symbols from (29), and recover the result of the direct calculation (12).

1.4 Geodesic Curves in Algebraic Form

In order to find the geodesic curves there are regular four steps:

1. Identify symmetries and find the integrals of motion arising from them.
2. Write explicitly the normalization constraint of the velocity vector.
3. Substitute the integrals of motion into the normalization constraint.
4. Either solve the first order ODE's to find the curves in parametric form $x^i(s)$, or divide the velocities and integrate to find the curves in algebraic form.

The **first step** is to find integrals of motion arising from symmetries.

The metric is independent of the x-coordinate, therefore a translation of the x coordinate is a symmetry. The associated *Killing vector field* is

$$\xi^i = (1, 0) \quad (30)$$

and the conserved quantity is

$$p = \xi \cdot u = g_{ij}\xi^i u^j = g_{xj}u^j = g_{xx}u^x = \frac{1}{y^2}\dot{x} \quad (31)$$

where we denote with dot $\dot{x} \equiv \frac{dx}{ds}$. This is the conserved canonical momentum $\frac{\partial L}{\partial\left(\frac{dx}{ds}\right)}$ we found in (20). p is called an integral of motion, it is constant along trajectories that satisfy the equations of motion, i.e., along geodesics.

The **second step** is to write the first integral, namely, the normalization constraint

$$u \cdot u = 1 \quad (32)$$

$$g_{ij}u^i u^j = \frac{1}{y^2}\dot{x}^2 + \frac{1}{y^2}\dot{y}^2 = 1 \tag{33}$$

The **third step** is to substitute the integral of motion (31) into the normalization constraint (33)

$$\dot{x} = py^2 \tag{34}$$

$$\dot{x}^2 = p^2 y^4 \tag{35}$$

$\Rightarrow$

$$\frac{1}{y^2}\left(p^2 y^4 + \dot{y}^2\right) = 1 \tag{36}$$

This is a first order ODE for y

$$\dot{y} = \pm y\sqrt{1 - p^2 y^2} \tag{37}$$

In order to find the algebraic form of the curve, the **fourth step** is to divide the velocities (34)/(37) (for $\dot{y} > 0$)

$$\frac{dx}{dy} = \frac{\frac{dx}{ds}}{\frac{dy}{ds}} = \frac{\dot{x}}{\dot{y}} = \frac{py}{\sqrt{1 - p^2 y^2}} \tag{38}$$

and integrate

$$x = \int \frac{dx}{dy} dy = \int \frac{py}{\sqrt{1 - p^2 y^2}} dy = -\sqrt{\frac{1}{p^2} - y^2} + x_0 \tag{39}$$

Therefore

$$(x - x_0)^2 + y^2 = \frac{1}{p^2} \tag{40}$$

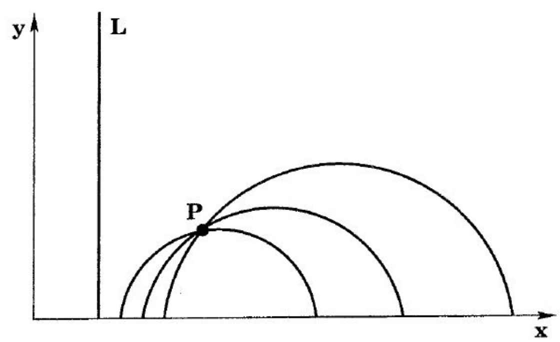
Half-circle geodesics

(40) are half-circles (since $y > 0$) centered on the x-axis at $(x_0, 0)$, with radius $\frac{1}{p}$. For $p = 0$ this infinite circle turn into a vertical line. Explicitly, we can plug $p = 0$ into (38) to find $\frac{dx}{dy} = 0$, thus

$$x = x_0 \tag{41}$$

Vertical line geodesics

Indeed, this geodesic line of constant x corresponds to motion with no momentum in the x direction $p = 0$.



Remark: *This example was important in the history of geometry. Euclid's fifth postulate for Euclidean geometry states that for a straight line L and a point P there is only one straight line (a geodesic) through P that does not intersect L. (That straight line is the one parallel to L.) The sphere is an example for which there are no such straight lines through P (all great circles intersect.) The hyperbolic plane is a constant negative curvature example (see Chapter 21) where there are an infinite number of straight lines through P that do not intersect L (see the example in the figure above).*

Figure 1: Geodesics of the Poincare half-plane, from Hartle

1.5 Geodesic Curves in Parametric Form

In order to find the parametric form of the curve, the **fourth step** is to solve (37)

$$\int ds = \int \frac{dy}{y\sqrt{1-p^2y^2}} \tag{42}$$

Wolfram says that

$$s = -\tanh^{-1}\left(\sqrt{1-p^2y^2}\right) \tag{43}$$

for simplicity we took the integration constant to be zero.

$$\tanh^2(s) = 1 - p^2y^2 \tag{44}$$

$\Rightarrow$

$$y = \frac{1}{p}\sqrt{1-\tanh^2(s)} \tag{45}$$

$\Rightarrow$

$$\boxed{y(s) = \frac{1}{p\cosh(s)}} \tag{46}$$

Parametric form of the geodesics

Plug into (34)

$$\frac{dx}{ds} = \frac{1}{p\cosh^2(s)} \tag{47}$$

$$\boxed{x(s) = \frac{1}{p}\tanh(s)} \tag{48}$$

Parametric form of the geodesics

$$\begin{aligned} x^2 + y^2 &= \frac{1}{p^2}\left(\tanh^2(s) + \frac{1}{\cosh^2(s)}\right) \\ &= \frac{1}{p^2}\left(\frac{\sinh^2(s)+1}{\cosh^2(s)}\right) = \frac{1}{p^2} \end{aligned} \tag{49}$$

These are the circles centered at the origin.

2 Lorentz Hyperboloid

1. Consider a 3-dimensional Minkowski space. Write the equations of all the "spheres" centered at the origin, in Cartesian coordinates (t, x, y). Draw them and classify the surfaces.

2. Find the coordinates transformations to hyperbolic-polar coordinates (a, χ, ϕ) for points inside the center light-cone (analouge of spherical-polar coordinates in Euclidean space).

3. Find the metric in these coordinates and the induced metric on a spacelike hyperboloid (sphere).

2.1 Spheres in Minkowski Space

We define a sphere as the set of points with the same quadrance from some center point. As for circles in Minkowski plane, there are three types of spheres Minkowski space:

Spheres with negative quadrance $q(\boldsymbol{x}) = -a^2$

$$-t^2 + x^2 + y^2 = -a^2 \tag{50}$$

These spheres look like *hyperboloids of two sheets* (inside the center light cone). The radius vector $\boldsymbol{x}$ have negative quadrance (it is timelike) and is orthogonal to the surface. All the tangent vectors have positive quadrance, i.e., they are spacelike. The surface is a spacelike surface.

Spheres with positive quadrance $q(\boldsymbol{x}) = a^2$

$$-t^2 + x^2 + y^2 = a^2 \tag{51}$$

These spheres look like *hyperboloids of one sheet* (outside the center light cone). The radius vector $\boldsymbol{x}$ have positive quadrance (it is spacelike) and is orthogonal to the surface. The tangent vectors have either positive or negative quadrance, i.e., they are spacelike or timelike. The surface is a timelike surface.

Spheres with zero quadrance $q(\boldsymbol{x}) = 0$

$$-t^2 + x^2 + y^2 = 0 \tag{52}$$

This is the center light-cone. It is a null surface.

2.2 Hyperbolic-Polar Coordinates

Inside the center light-cone the radial coordinate a is

$$-t^2 + x^2 + y^2 = -a^2 \tag{53}$$

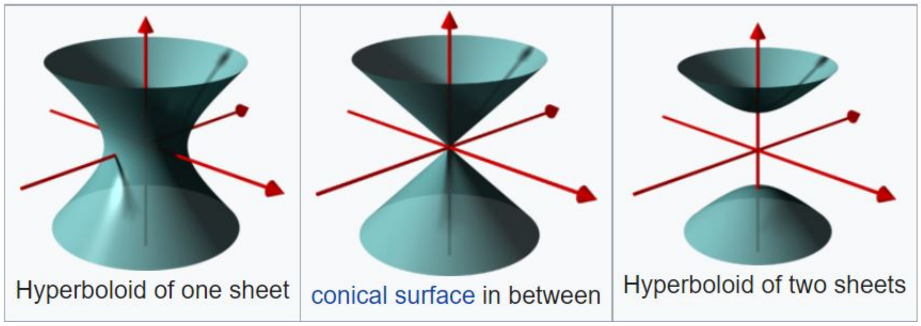


Figure 2: Hyperboloids

We define the radial coordinate from the t axis

$$\rho^2 = x^2 + y^2 \tag{54}$$

and transform the spatial coordinates to polar coordinates (ρ, ϕ)

$$\begin{aligned} x &= \rho \cos \phi \\ y &= \rho \sin \phi \end{aligned} \tag{55}$$

The hyperboloids equations now have the form

$$-t^2 + \rho^2 = -a^2 \tag{56}$$

Now use the hyperbolic angle χ from the t axis

$$t = a \cosh \chi \tag{57}$$

$$\rho = a \sinh \chi \tag{58}$$

The total coordinates transformations are

Hyperbolic-polar coordinates

$$t = a \cosh \chi \tag{59}$$

$$x = a \sinh \chi \cos \phi \tag{60}$$

$$y = a \sinh \chi \sin \phi \tag{61}$$

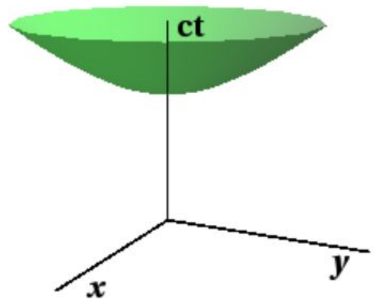


Figure 3: Lorentz Hyperboloid

2.3 The Metric and the Induced Metric

The metric of Minkowski space in Cartesian coordinates (t, x, y) is

$$ds^2 = -dt^2 + dx^2 + dy^2 \tag{62}$$

In the cylindrical coordinates (t, ρ, ϕ) the metric is

$$ds^2 = -dt^2 + d\rho^2 + \rho^2 d\phi^2 \tag{63}$$

The metric in hyperbolic polar coordinates (a, χ, ϕ) is

$$ds^2 = -da^2 + a^2 d\chi^2 + (a \sinh \chi)^2 \, d\phi^2 \tag{64}$$

$$ds^2 = -da^2 + a^2 \left(d\chi^2 + \sinh^2 \chi d\phi^2 \right) \tag{65}$$

Metric of Minkowski space

This is still a metric of Minkowski space, analogue to spherical polar coordinates in Euclidean 3-dim. space. a is the radial time coordinate, each constant a corresponds to a hyperbola in the t, ρ plane, which is a spacelike hyperboloid surface (a sphere) of quadrance $-a^2$. $a \sinh \chi = \rho$ is the distance from the t axis. We wanted to emphesize that together with ϕ, the hyperbola in the t, ρ plane becomes a surface of revolution around the t axis. This surface is a 2-dim. Lorentz hyperboloid. The induced metric on it is $(da = 0)$

$$ds^2_{hyperboloid} = a^2 \left(d\chi^2 + \sinh^2 \chi d\phi^2 \right) \tag{66}$$

2-dim. Lorentz hyperboloid metric

The two-dim. metric (66) is analogue to a metric on a sphere with radius

a. The difference is that χ is a hyperbolic angle, and there is $\sinh\chi$ factor. Compare to a plane of constant t (63) and a sphere (which is embedded in a 3-dim. **Euclidean** space):

$$ds^2_{plane} = d\rho^2 + \rho^2 d\phi^2 \tag{67}$$

$$ds^2_{sphere} = a^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{68}$$

The plane and sphere are also surfaces of revolution around the z axis, with distances from the axis of ρ and $a\sin\theta$ respectively.

Claim: The Lorentz hyperboloid is an embedding of the hyperbolic plane in Minkowski space! Like for the Euclidean sphere, the geodesics are the intersections of the hyperboloid with planes through the origin.

3 The Beltrami-Poincare Disc

There are five main (equivalent) models for hyperbolic geometry: The Klein disc, the Poincare disc, the Poincare half-plane, the Lorentz hyperboloid, and the hemisphere.

In the last sections we explored some aspects of the Poincare half-plane model and wrote a metric on the Lorentz hyperboloid. We would like to take a peak on their connection. This goes most naturally via a stereographic projection of the Lorentz hyperboloid to the *Poincare disc*.

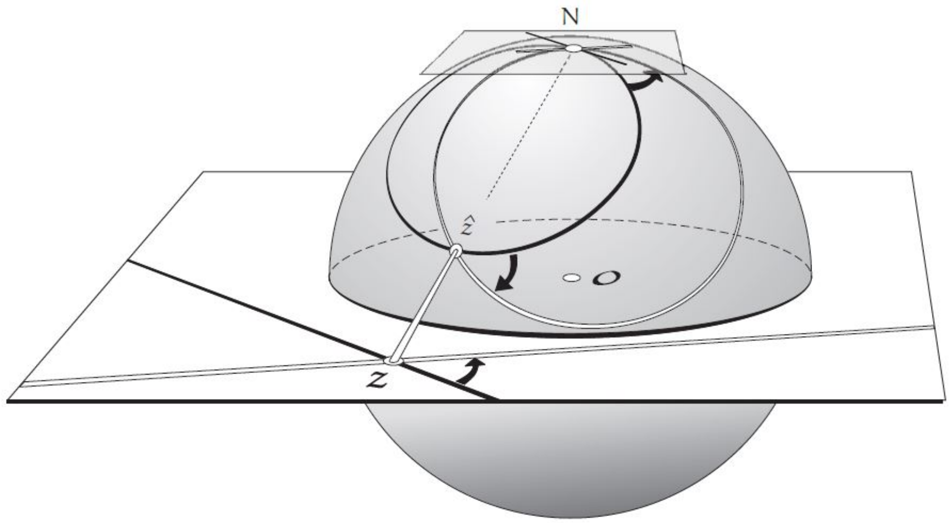
3.1 Stereographic Projection of a Sphere

3.1.1 Euclidean Sphere

We will find the metric of the sterographic map in a short and clever **geometric way**[1]. See Figure 4. Consider a sphere of radius a. Imagine a point source of light at the north pole N and project onto a plane that passes through the equator. The image of the point $\hat{z}$ on the sphere is the point z in the plane. Geometric facts about the projection:

1. The southern hemisphere is mapped to the interior of the circle of radius R; each point on the equator is mapped to itself; the northern hemisphere is mapped to the exterior of the circle, except N which is mapped to a "point at infinity" in any direction.

[1] Thanks to Needham.



[4.9] Stereographic Projection Is Conformal. *As the line rotates around z, the tangent at N of its circular image rotates with it, so the angles at z and N are equal. But, by symmetry, the angle at ẑ equals the angle at N, so this equals the angle at z, proving that stereographic projection is conformal.*

Figure 4: Sterographic projection of a Euclidean sphere

2. The stereographic image of a line in the plane is a circle on the sphere that passes through the north pole. This is because as z moves along a line, the line connecting it to N sweeps out a part of a plane through N.

3. The direction of the circle on the sphere at N is parallel to the original line. This is because the tangent plane at N is parallel to the equatorial plane, and if we slice through two parallel planes with a third plane, we obtain two parallel lines of intersection.

4. **The stereographic projection preserves angles (it is conformal)**. This is because the tangents to the circles at N are parallel to the lines, and the angle of intersection between the two circles is the same at their two intersection points, N and $\hat{z}$.

The fact that the map is conformal means that it stretches infinitesimal distances at some point with the same amount in any direction. The metric has the form

$$ds^2 = \Lambda\left(x, y\right)^2 \left(dx^2 + dy^2\right) \tag{69}$$

where x, y are Cartesian coordinates of the planar map. The line element is

$$ds = \Lambda(x, y)\, ds_E \tag{70}$$

where $ds_E = \sqrt{dx^2 + dy^2}$. We need to find the "stretching/squeezing" factor $\Lambda(x, y)$. We can look at a line element on the sphere in any direction, so we choose to orient it along some latitude. We solve in three steps:

1. ds and ds_E are both arc lengths of the same angle from N (rotate a little Figure 5.), thus they are in proportion

$$\frac{ds}{ds_E} = \frac{N\hat{z}}{Nz} \tag{71}$$

2. An inscribed angle resting on a diameter is $\frac{\pi}{2}$. The triangles $\Delta N\hat{z}S$ and ΔNOz are similar. Therefore the proportions

$$\frac{2R}{N\hat{z}} = \frac{Nz}{R} \tag{72}$$

   therefore

$$ds = \frac{2R^2}{(Nz)^2} ds_E \tag{73}$$

3. Use Pythagoras theorem for Triangle ΔNOz

$$(Nz)^2 = R^2 + r^2 \tag{74}$$

   where $r^2 = x^2 + y^2$.

Now we have

$$ds = \frac{2R^2}{R^2 + r^2} ds_E \tag{75}$$

and the metric of the sphere in sterographic polar coordinates is

$$ds^2 = \frac{4R^4}{(R^2 + r^2)^2}\left(dr^2 + r^2 d\theta^2\right) \tag{76}$$

Metric of the sphere in sterographic polar coordinates

3.1.2 Minkowskian Sphere

Inspired by this derivation, since the projection is still conformal, we do it now in Minkowski space to obtain the Poincare disc metric[2]. See Figure 6. For the

[2] This is my original derivation.

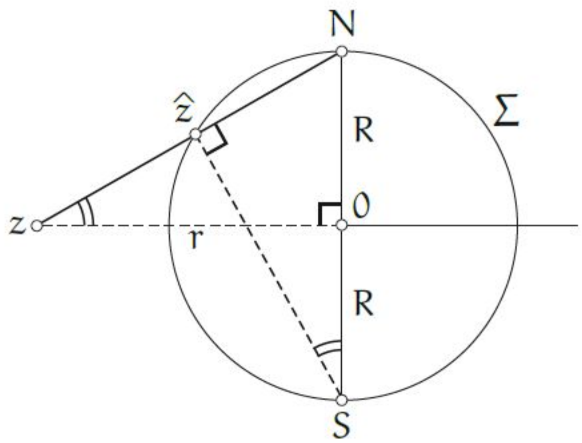


Figure 5: Slice of the sphere

hyperboloid we project from the south pole S.

1. ds and ds_E are both arc lengths of the same angle from S, thus they are in proportion
$$\frac{ds}{ds_E} = \frac{S\hat{z}}{Sz} \tag{77}$$

2. We use the generalized theorem: A triangle inscribed in a circle with the diameter as one of the edges has the two other edges orthogonal to one another. Thus, the triangles $\Delta S\hat{z}N$ and ΔSOz are similar, and
$$\frac{2R}{S\hat{z}} = \frac{Sz}{R} \tag{78}$$
therefore
$$ds = \frac{2R^2}{(Sz)^2} ds_E \tag{79}$$

3. Use Minkowskian Pythagoras theorem for Triangle ΔSOz
$$-(Sz)^2 = -R^2 + r^2 \tag{80}$$
where $r^2 = x^2 + y^2$.

Now we have
$$ds = \frac{2R^2}{R^2 - r^2} ds_E \tag{81}$$

and the metric of Poincare disc in polar coordinates is

$$ds^2 = \frac{4R^4}{\left(R^2 - r^2\right)^2} \left(dr^2 + r^2 d\theta^2\right) \tag{82}$$

Metric of Poincare disc in polar coordinates

See in Figure 7 the geodesics, and that a triangle has **less** then π radians.

3.2 Back to the Half-Plane

The transformation from Poincare disc Cartesian coordinates (x', y') to Poincare half-plane Cartesian coordinates (x, y) reads

$$(x, y) = R \left(\frac{2Rx'}{x'^2 + (R - y')^2}, \frac{R^2 - x'^2 - y'^2}{x'^2 + (R - y')^2} \right) \tag{83}$$

Geometrically, we hold the disc at one point on the boundary $(x' = 0, y' = R)$, remove it, and open up the disc and stretch it such that the circumference boundary lie on the horizontal $y = 0$ axis (plug in $x'^2 + y'^2 = R^2$). The result is the upper half-plane. Recall that the points on boundary of the disc were at infinite distance, and now they are on the x-axis, while the one point on the boundary that we took out was sent to $y \to \infty$. Most of the geodesic lines on the disc are transformed to half circles orthogonal to the x-axis in the half plane, but the geodesics that were connected to the point we took out become vertical lines.

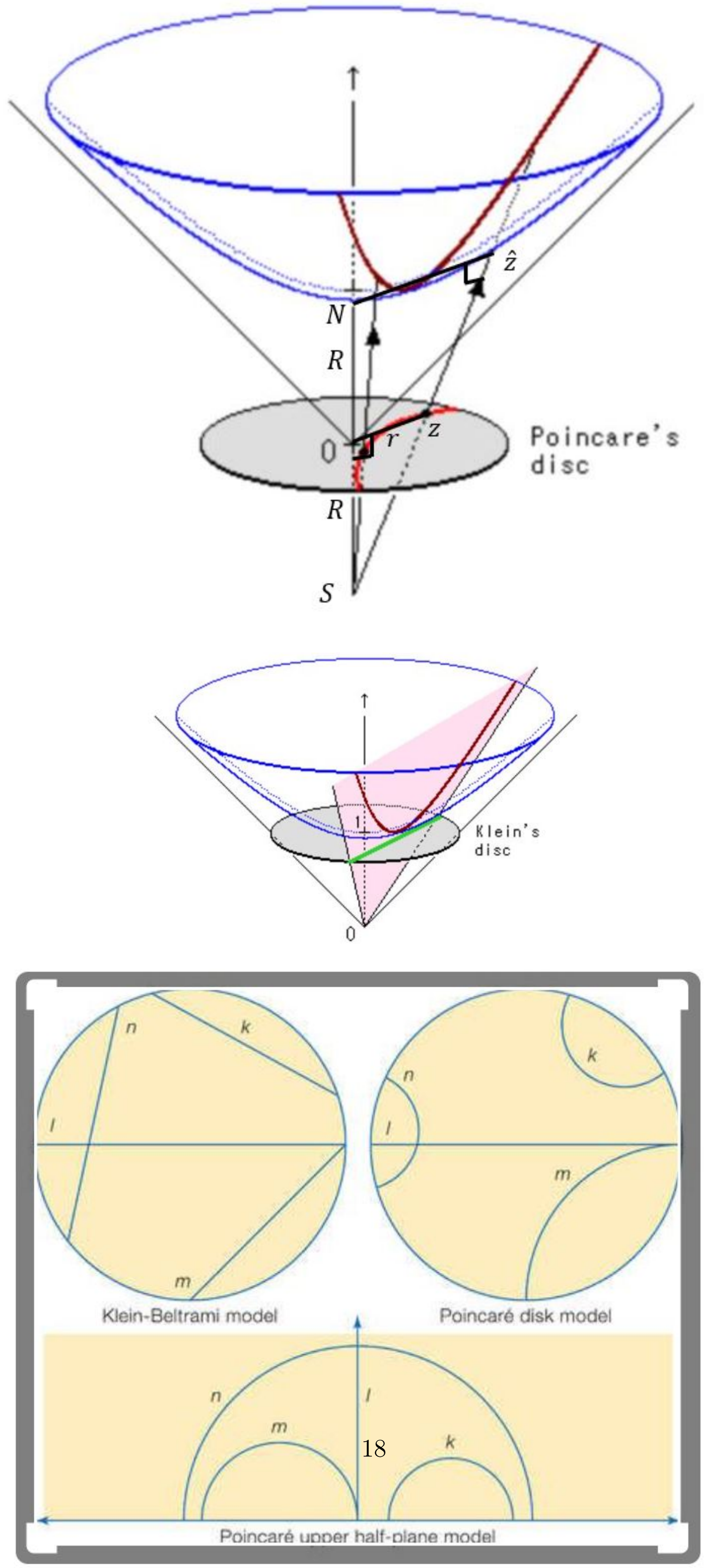
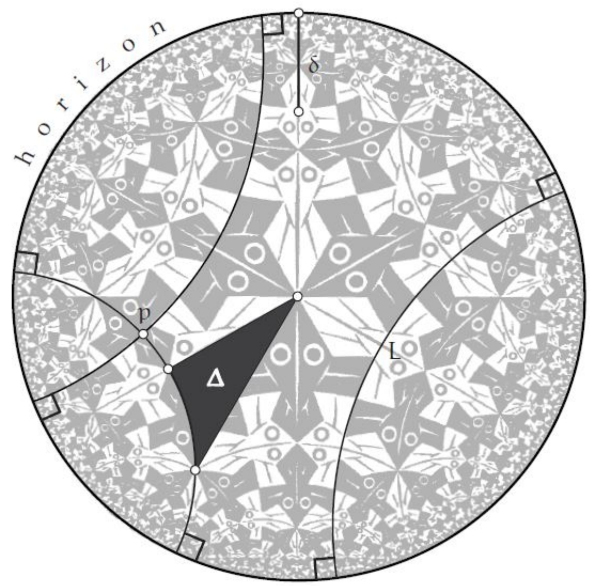


Figure 6: Projections and geodesics of different models of the hyperbolic plane



[5.11] *The Beltrami–Poincaré disc model of the hyperbolic plane. The background is Escher's* Circle Limit I*; superimposed are hyperbolic lines, which are diameters and circular arcs orthogonal to the infinitely distant boundary circle (the horizon). Clearly the Hyperbolic Axiom (1.1) is satisfied, and* $\mathcal{E}(\Delta) < 0$. *M. C. Escher's* Circle Limit I *© 2020 The M. C. Escher Company-The Netherlands. All rights reserved. www.mcescher.com.*

[5.12] *Escher's* Circle Limit I *transformed (by Professor John Stillwell) from its original conformal disc model [5.11] to the conformal half-plane model. M. C. Escher's* Circle Limit I *© 2020 The M. C. Escher Company-The Netherlands. All rights reserved. www.mcescher.com.*

Figure 7: Escher and hyperbolic geometry