כבידה 1

2022/23 סמסטר א'

פתרון תרגיל 4

1. מערכת ייחוס מאיצה

a) The line element, shown in part (a) of the solution to Problem 6, is independent of t'. So the spatial distance $\Delta x'$ between two curves of constant x' remains constant in t'.

b) The four-velocity in the (t, x) frame can be calculated from the expression for $x(t')$ and $y(t')$ given in Problem 6, together with the line element derived in part (a) of that problem which gives the connection between t' and proper time τ along a curve of constant x', y', and z', viz.,

$$d\tau = \left(1 + \frac{gx'}{c^2}\right) dt' .$$

The resulting four-velocity of a curve at constant x' has components

$$\frac{dt}{d\tau} = \cosh\left(\frac{gt'}{c}\right) , \frac{dx}{d\tau} = c \sinh\left(\frac{gt'}{c}\right) .$$

(Components in the y- and z-direction vanish). The components of the four-acceleration are

$$a^t = \frac{d^2t}{d\tau^2} = \frac{g}{c}\left(1 + \frac{gx'}{c^2}\right)^{-1} \sinh\left(\frac{gt'}{c}\right) ,$$
$$a^x = \frac{d^2x}{d\tau^2} = g\left(1 + \frac{gx'}{c^2}\right)^{-1} \cosh\left(\frac{gt'}{c}\right) .$$

Thus,

$$a = (\mathbf{a} \cdot \mathbf{a})^{\frac{1}{2}} = \frac{g}{(1 + gx'/c^2)}$$

which decreases with x'.

2. סנכרון שעונים

א. הפוטנציאל במקרה זה הוא פשוט $\phi = gh$ ולכן עבור צופה B:

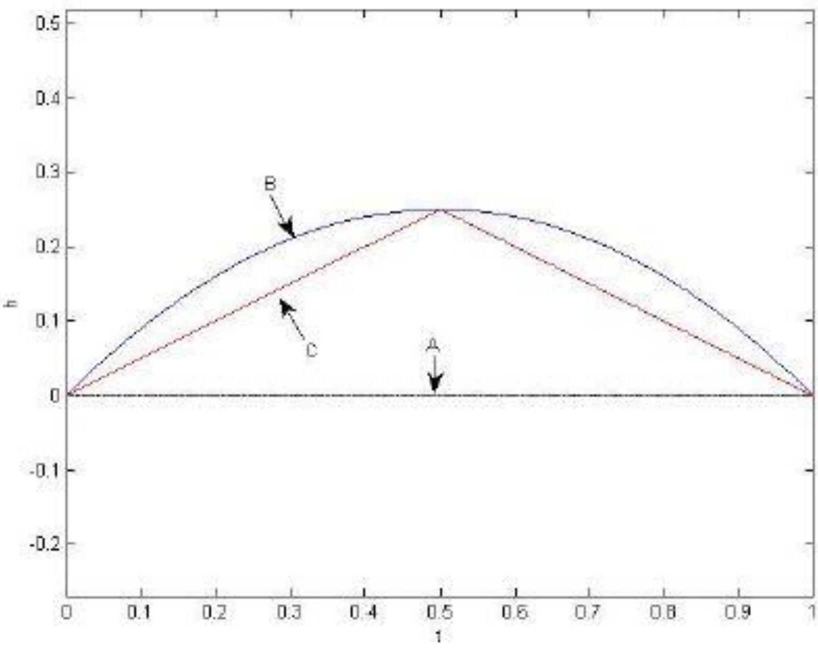
$$T_B = \int_0^T dt(1+gh_B) = \int_0^T dt\left(1+g\cdot\frac{1}{2}gt(T-t)\right)$$
$$= T + \frac{1}{4}g^2T^3 - \frac{1}{6}g^2T^3 = T\left(1+\frac{1}{12}g^2T^2\right)$$

עבור צופה C:

$$T_B = \int_0^T dt(1+gh_C) = T + \int_0^{T/2} dtg\frac{gT}{4}t + \int_{T/2}^{T} dtg\frac{gT}{4}(T-t)$$
$$= T + 2\cdot\frac{1}{8}g^2T\left(\frac{T}{2}\right)^2 = T\left(1+\frac{1}{16}g^2T^2\right)$$

שעונו של צופה B מודד את הזמן הארוך ביותר!

ב. אם נשרטט את מסלולי השעונים:



קל לראות שצופה B שוהה הכי רחוק ממוקד המשיכה ועל כן לפי הכלל שככל שהשעון קרוב יותר למוקד כך הוא מודד זמן קצר יותר, שעונו של B צריך למדוד זמן ארוך יותר.

ג. במקרה זה יש להכליל את הנוסחה ל- $dt' = \left(1+\phi-\frac{1}{2}v^2\right)dt$:

עבור צופה B:

$$T_B = \int_0^T dt\left(1 + gh_B - \frac{1}{2}v_B^2\right) = \int_0^T dt\left(1 + g\cdot\frac{1}{2}gt(T-t) - \frac{1}{2}g^2\left(\frac{T}{2}-t\right)^2\right)$$

$$= T\left(1+\frac{1}{12}g^2T^2\right) - \frac{1}{6}g^2\cdot 2\cdot\left(\frac{T}{2}\right)^3 = T\left(1+\frac{1}{24}g^2T^2\right)$$

עבור צופה C:

$$T_B = \int_0^T dt\left(1 + gh_C - \frac{1}{2}v_C^2\right)$$

$$= T + \int_0^{T/2} dtg\frac{gT}{4}t + \int_{T/2}^T dtg\frac{gT}{4}(T-t) - \frac{1}{2}\int_0^T dt\left(\frac{gT}{4}\right)^2$$

$$= T\left(1+\frac{1}{16}g^2T^2\right) - \frac{1}{32}g^2T^3 = T\left(1+\frac{1}{32}g^2T^2\right)$$

3. ערפילית כדורית

ו. נחשב את הפוטנציאל ממשוואת פואסון:

$$\frac{1}{r^2}\left(r^2\phi'\right)' = 4\pi G\rho \Rightarrow \phi = 4\pi\int\frac{dr}{r^2}\int_0^r dr' r'^2 G\rho$$
$$= \frac{4\pi G\rho}{3}\int\frac{dr}{r^2}r^3 = \frac{2\pi}{3}G\rho r^2$$

ז. נשתמש בנוסחת השדה החלש:

$$\frac{\omega}{\omega'} = 1+\phi = 1+\frac{2\pi G}{3}\rho R^2$$

ח. נחשב את הצפיפות האפקטיבית של השמש:

$$\rho = \frac{M}{\frac{4\pi}{3}R^3} \approx \frac{2\cdot 10^{30}}{\frac{4\pi}{3}\left(5\cdot 10^8\right)^3} {kg}/{m^3} \approx 3.8\cdot 10^3\, {kg}/{m^3}$$

נציב את הנתון להסחה לאדום:

$$\frac{\omega}{\omega'} - 1 = \frac{2\pi G}{3}\frac{\rho R^2}{c^2} \approx \frac{2\pi\cdot 6.67\cdot 10^{-11}}{3}\cdot\frac{3.8\cdot 10^3\cdot\left(5\cdot 10^8\right)^2}{\left(3\cdot 10^8\right)^2} \approx 1.5\cdot 10^{-6}$$

ט. יש לדרוש שההסחה לאדום תהיה קטנה בהרבה מ-1:

$$\frac{2\pi G}{3}\rho R^2 << 1 \Rightarrow R << R_C = \sqrt{\frac{3}{2\pi G\rho}}$$

$$\frac{v^2}{r} = \frac{d\phi}{dr} = \frac{4\pi G\rho}{3}r = \frac{2\phi}{r} \Rightarrow v^2 = 2\phi$$

ולכן הזמן העצמי שלו מקיים:

$$d\tau^2 = \left(1+2\phi - v^2\right)dt^2 = dt^2$$

כלומר, הוא מסונכרן עם צופה במרכז הענן!

4. מערכת מסתובבת

$$
\begin{aligned}
ds^2 &= -\left[1-\Omega^2\left(x^2+y^2\right)\right]dt^2+2\Omega\left(ydx-xdy\right)dt+ \\
&\quad +dx^2+dy^2+dz^2 \\
x &= r\,sin\theta cos\phi \\
dx &= sin\theta\,cos\phi\,dr+cos\theta\,cos\phi\,r\,d\theta-sin\theta\,sin\phi\,r\,d\phi \\
y &= r\,sin\theta\,sin\phi \\
z &= r\,cos\,theta \\
dx^2+dy^2+dz^2 &= dr^2+r^2d\theta^2+r^2sin^2\theta d\phi^2 \\
x^2+y^2 &= r^2sin^2\theta.
\end{aligned}
$$

$$
\begin{aligned}
ydx-xdy &= r\,sin\theta\{sin\theta cos\phi sin\phi dr+cos\theta cos\phi sin\phi rd\theta- \\
&\quad -sin\theta sin^2\phi-sin\theta cos\phi sin\phi dr- \\
&\quad -rd\theta cos\theta cos\phi sin\phi-sin\theta cos^2\phi rd\phi\}
\end{aligned}
$$

The cosines all cancel out and we remain with

$$ydx-xdy=-r^2d\phi sin^2\theta.$$

Now we work out the line element:

$$
\begin{aligned}
ds^2 &= -dt^2+\Omega^2r^2sin^2\theta dt^2-2\Omega r^2sin^2\theta dtd\phi+ \\
&\quad +r^2sin^2\theta d\phi^2+dr^2+r^2d\theta^2 \\
&= -dt^2+r^2sin^2\theta\left(\Omega dt-d\phi\right)^2+dr^2+r^2d\theta^2.
\end{aligned}
$$

We take $\phi\rightarrow\phi'=\phi-\Omega t$ as the question says, and check the line element. Then $d\phi'=d\phi-\Omega dt$, and we get

$$ds^2=-dt^2+r^2sin^2\theta d\phi'^2+r^2d\theta^2+dr^2$$

(b) To find the geodesic equations we need to minimize the Lagrangian:

$$
\begin{aligned}
L &= [\left(1-\Omega^2\left(x^2+y^2\right)\right)\left(\frac{dt}{d\sigma}\right)^2-2\Omega\left(y\frac{dx}{d\sigma}-x\frac{dy}{d\sigma}\right)\frac{dt}{d\sigma}- \\
&\quad -\left(\frac{dx}{d\sigma}^2\right)-\left(\frac{dy}{d\sigma}^2\right)-\left(\frac{dz}{d\sigma}^2\right)]^{1/2}
\end{aligned}
$$

The E-L equation for x:

$$\begin{aligned}
\frac{d}{d\sigma}\frac{\partial L}{\partial \frac{dx}{d\sigma}} - \frac{\partial L}{\partial x} &= 0 \\
\frac{\partial L}{\partial x} &= -\frac{x\Omega^2}{L}\left(\frac{dt}{d\sigma}\right)^2 + \Omega\frac{1}{L}\frac{dy}{d\sigma}\frac{dt}{d\sigma} \\
\frac{\partial L}{\partial \frac{dx}{d\sigma}} &= -\Omega y\frac{dt}{d\sigma}\frac{1}{L} - \frac{1}{L}\frac{dx}{d\sigma} = -\Omega\frac{dt}{d\tau} - \frac{dx}{d\tau}
\end{aligned}$$

Now $Ld\sigma = d\tau$ and so

$$\begin{aligned}
\frac{1}{L}\frac{d}{d\sigma}\frac{\partial L}{\partial \frac{dx}{d\sigma}} &= -\frac{d^2x}{d\tau^2} - \frac{d}{d\tau}\left(\Omega y\frac{dt}{d\tau}\right) \\
&= \frac{1}{L}\frac{\partial L}{\partial x} = -x\Omega^2\left(\frac{dt}{d\tau}\right)^2 + \Omega\frac{dy}{d\tau}\frac{dt}{d\tau} \\
-\frac{d^2x}{d\tau^2} - \Omega y\frac{d^2t}{d\tau^2} &= 2\Omega\frac{dy}{d\tau}\frac{dt}{d\tau} - x\Omega^2\left(\frac{dt}{d\tau}\right)^2
\end{aligned}$$

This looks more complicated and different than what we did in class, that is because if you look at the metric you will see the components are more complicated: g_{tt} is a function of x and y, and you also have non diagonal components (g_{xt}, g_{yt}). For z it's simple:

$$\frac{d^2z}{d\tau^2} = 0$$

(c) In the non relativistic limit

$$\begin{aligned}
\frac{dt}{d\tau} &= 1, \frac{d^2t}{d\tau^2} = 0, \\
\frac{d^2x}{dt^2} &= 2\Omega\frac{dy}{dt} - x\Omega^2
\end{aligned}$$

where the first term is the Coriolis force and the second centrifugal.