Gravity 1 - Tutorial 7

Measurements

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1 Coordinate Basis and Orthonormal Basis

1.1 Coordinate Basis

Definition: Velocity vector **components** in the *coordinate basis* are the **coordinate velocities**.

The expansion of the vector U in the coordinate basis e_μ is

$$U = U^\mu e_\mu \tag{1}$$

where

$$U^\mu = \frac{dx^\mu}{d\lambda} \tag{2}$$

for some parameter λ. In general the parameter is arbitrary, and this basis does not involve a metric in its definition. However, given a metric, we like to write velocities with the arc length parameter τ, so that

$$U^\mu = \frac{dx^\mu}{d\tau} \tag{3}$$

and U is normalized

$$g(U,U) = g(U^\mu e_\mu, U^\nu e_\nu) = g(e_\mu, e_\nu) U^\mu U^\nu = g_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -1 \tag{4}$$

for a timelike vector.

We used the relation to the metric

$$e_\mu(x) \cdot e_\nu(x) = g_{\mu\nu}(x) \tag{5}$$

A coordinate velocity is the rate of change of the coordinate, like angular velocity for example. It has units of 1/time.

The magnitude of the μ basis vector is

$$|e_\mu| = \sqrt{g(e_\mu, e_\mu)} = \sqrt{g_{\mu\mu}} \tag{6}$$

Magnitude of coordinate basis vector

It is not unity in general.

What is the geometric meaning of the **direction** of the coordinate basis?

A coordinate basis vector e_χ is pointing in the direction of increasing coordinate χ. The coordinate lines can be orthogonal or not.

What is the geometric meaning of the **magnitude** of the coordinate basis?

Suppose we increase some coordinate χ from χ_1 to χ_2, while keeping all the other fixed. We travel along some curve, and the arc length we pass is given by the metric

$$\Delta s = \int ds = \int_{\chi_1}^{\chi_2} \sqrt{g_{\chi\chi}} d\chi = \int_{\chi_1}^{\chi_2} |e_\chi| d\chi \tag{7}$$

So, the magnitude of the basis vector e_χ is the infinitesimal distance we travel in the real world, along a curve of varying coordinate χ, when we increase coordinate χ by $d\chi$. In the case that $g_{\chi\chi}$ is independent of χ then $|e_\chi|$ is just the finite length we travel when increasing χ by 1.

1.2 Orthonormal Basis

Definition: *Orthonormal basis* at a point is a set of d orthonormal vectors.

1. The definition is in the name, but few remarks are in order:
   Unlike the coordinate basis, this basis is only defined by a metric, which measures lengths and angles.

2. A priori the orthonormal basis has nothing to do with coordinates. Nevertheless, if we use "orthogonal" coordinates it is convenient to choose an orthonormal basis in the directions of the coordinate lines.

3. At each point there is a different vector space and different orthonormal basis.

The expansion of the vector U in the orthonormal basis $e_{\hat{\mu}}$ is

$$U = U^{\hat{\mu}} e_{\hat{\mu}} \tag{8}$$

where

$$|e_{\hat{\mu}}| = \sqrt{g\left(e_{\hat{\mu}}, e_{\hat{\mu}}\right)} = \sqrt{g_{\hat{\mu}\hat{\mu}}} = 1 \tag{9}$$

(1,-1, or 0, for timelike/spacelike/null vectors).

The metric components at each point in orthonormal basis are

$$g_{\hat{\mu}\hat{\nu}} = e_{\hat{\mu}} \cdot e_{\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}} \tag{10}$$

for a Lorentzian manifold (signature $(-1, 1, ...)$), and

$$g_{\hat{i}\hat{j}} = e_{\hat{i}} \cdot e_{\hat{j}} = \delta_{\hat{i}\hat{j}} \tag{11}$$

for a Riemannian manifold (signature $(1, 1, ...)$).

In the case where we have orthogonal coordinates, we only need to normalize the coordinate basis vectors, in order to create an orthonormal basis.

$$e_{\hat{\mu}} = \frac{e_\mu}{|e_\mu|} = \frac{e_\mu}{\sqrt{g_{\mu\mu}}} \tag{12}$$

where we used (6). Say that χ is the second coordinate, then the components of e_χ in the coordinate basis are

$$(e_\chi)^\mu = (0, 1, 0, 0) \tag{13}$$

and the components of $e_{\hat{\chi}}$ in the coordinate basis are

$$\boxed{(e_{\hat{\chi}})^\mu = \left(0, g_{\chi\chi}^{-\frac{1}{2}}, 0, 0\right)} \tag{14}$$

Indeed, (14) is normalized. Compute its magnitude with by the coordinate basis

$$|e_{\hat{\chi}}|^2 = g_{\mu\nu} \left(e_{\hat{\chi}}\right)^\mu \left(e_{\hat{\chi}}\right)^\nu = g_{\chi\chi} \left(\left(e_{\hat{\chi}}\right)^\chi\right)^2 = g_{\chi\chi} \left(g_{\chi\chi}^{-\frac{1}{2}}\right)^2 = 1 \tag{15}$$

Remark: For the timelike basis vector

$$\boxed{(e_{\hat{0}})^\mu = \left(\left(-g_{00}\right)^{-\frac{1}{2}}, 0, 0, 0\right)} \tag{16}$$

.

1.3 Example: Polar Coordinates

The common choice of basis when working with polar coordinates is the orthonormal basis, where $e_{\hat{r}} \equiv \hat{r}$ is the radial basis vector and $e_{\hat{\theta}} \equiv \hat{\theta}$ is the tangential basis vector. The velocity vector is expanded as

$$U = U^{\hat{r}} e_{\hat{r}} + U^{\hat{\theta}} e_{\hat{\theta}} \tag{17}$$

where

$$U^{\hat{r}} = \frac{dr}{dt} \tag{18}$$

is the radial velocity and

$$U^{\hat{\theta}} = r\frac{d\theta}{dt} \tag{19}$$

is the tangential velocity. Of course $e_{\hat{r}} \cdot e_{\hat{r}} = e_{\hat{\theta}} \cdot e_{\hat{r}} = 1$ and $e_{\hat{r}} \cdot e_{\hat{\theta}} = 0$. Polar coordinates lines are orthogonal, and this orthonormal basis is oriented such to point in their directions. Let us transform to the coordinate basis by rewriting (17) as

$$\begin{aligned} U &= \frac{dr}{dt} e_{\hat{r}} + \frac{d\theta}{dt} \left(r e_{\hat{\theta}} \right) \\ &= U^r e_r + U^\theta e_\theta \end{aligned} \tag{20}$$

where

$$e_r = e_{\hat{r}} \tag{21}$$

and

$$e_\theta = r e_{\hat{\theta}} \tag{22}$$

since

$$U^r = \frac{dr}{dt} \qquad U^\theta = \frac{d\theta}{dt} \tag{23}$$

So in this simple example the difference between the usual orthonormal basis and the coordinate basis is to whom the factor r belongs to. In the orthonormal basis the basis $e_{\hat{\theta}}$ must be unit length, so the component (19) has it. In the coordinate basis the component U^θ is the coordinate velocity, i.e., the angular velocity, so the angular basis vector scales as r. From (22) it follows

$$|e_\theta| = |r||e_{\hat{\theta}}| = r \tag{24}$$

Lets see how it relate to the metric. The metric in polar coordinates reads

$$ds^2 = dr^2 + r^2 d\theta^2 \tag{25}$$

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \tag{26}$$

$$g_{rr} = 1 \qquad g_{\theta\theta} = r^2 \qquad g_{r\theta} = 0 \tag{27}$$

Let us now transform to the orthonormal basis by denoting

$$dl = r d\theta \tag{28}$$

$$ds^2 = dr^2 + dl^2 \tag{29}$$

$$g_{\hat{i}\hat{j}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{30}$$

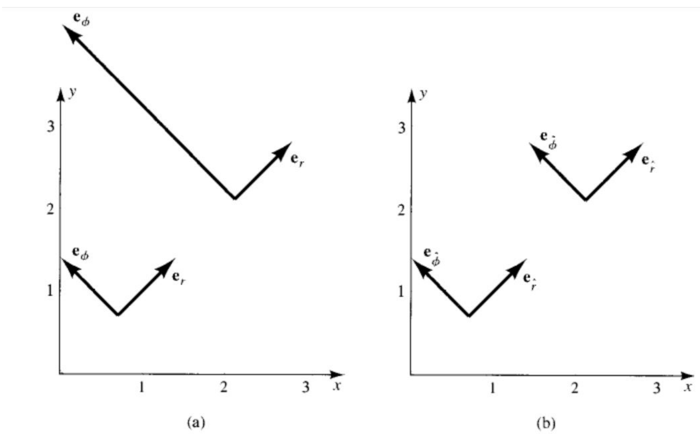

FIGURE 7.7 Coordinate and orthonormal basis vectors for polar coordinates in the plane. At left, the coordinate basis vectors point along the coordinate lines and have lengths $|\mathbf{e}_r| = 1$, $|\mathbf{e}_\phi| = r$. At right, the orthonormal basis vectors shown also point along the same coordinate lines but have unit length.

Figure 1: Coordinate and orthonormal polar bases, from Hartle

$$g_{\hat{r}\hat{r}} = 1 \qquad g_{\hat{\theta}\hat{\theta}} = 1 \qquad g_{\hat{r}\hat{\theta}} = 0 \tag{31}$$

We replaced the angle coordinate by the arc length along a constant r. From (26) and (30)

$$|e_\theta| = \sqrt{g\left(e_\theta, e_\theta\right)} = \sqrt{g_{\theta\theta}} = r \tag{32}$$

$$|e_{\hat{\theta}}| = \sqrt{g\left(e_{\hat{\theta}}, e_{\hat{\theta}}\right)} = \sqrt{g_{\hat{\theta}\hat{\theta}}} = 1 \tag{33}$$

Recall the geometric meaning of $|e_\theta|$. Traveling on a curve of constant r,

$$\Delta s = \int ds = \int dl = \int_{\theta_1}^{\theta_2} r d\theta = r\Delta\theta \tag{34}$$

The magnitude of the coordinate basis vector $|e_\theta|$ is the arc length along the curve of constant r when we increase the coordinate θ by 1.

2 Measurements

Definition: An *observer* is traveling along a timelike worldline **and** is equipped with an orthonormal basis $\{e_{\hat{0}}, e_{\hat{1}}, e_{\hat{2}}, e_{\hat{3}}\}$, such that the timelike basis vector $e_{\hat{0}}$ is tangent to its worldline, i.e., $e_{\hat{0}} = u_{ob}$.

Suppose an observer and a particle meet, i.e., their worldlines intersect.

If the particle has a 4-momentum p, then the energy of the particle that the observer measures is

Measured energy

$$E = -p \cdot u_{ob} \tag{35}$$

since,

$$(u_{ob})^{\hat{\mu}} = (e_{\hat{0}})^{\hat{\mu}} = (1, 0, 0, 0) \tag{36}$$

$$-p \cdot u_{ob} = -p \cdot e_{\hat{0}} = -\eta_{\hat{\mu}\hat{\nu}} p^{\hat{\mu}} (e_{\hat{0}})^{\hat{\nu}} = p^{\hat{0}} = E \tag{37}$$

and the dot product of vectors is independent of the basis.

The measured magnitude of the 3-velocity (the speed) can be found then by

Relation between measured energy and speed

$$E = m\gamma \tag{38}$$

Remark: $p^t/m = \frac{dt}{d\tau}$ in general coordinates, but $\frac{dt}{d\tau} = \gamma = \left(1 - \boldsymbol{v}^2\right)^{-\frac{1}{2}}$ in an orthonormal basis.

Have another look, slowly: The components of the 4-momentum p in the othonormal basis are

$$p^{\hat{\mu}} = (E, \boldsymbol{P}) \tag{39}$$

i.e.,

$$p = E e_{\hat{0}} + P^{\hat{i}} e_{\hat{i}} \tag{40}$$

The measured energy is

$$\begin{aligned} -p \cdot u_{ob} = -p \cdot e_{\hat{0}} &= -\left(E e_{\hat{0}} + P^{\hat{i}} e_{\hat{i}}\right) \cdot e_{\hat{0}} \\ &= -E \left(\underbrace{e_{\hat{0}} \cdot e_{\hat{0}}}_{-1}\right) - P^{\hat{i}} \underbrace{e_{\hat{i}} \cdot e_{\hat{0}}}_{0} = E \end{aligned} \tag{41}$$

as we saw in (35). The measured momenta are

Measured momentum

$$P^{\hat{i}} = p \cdot e_{\hat{i}} \tag{42}$$

Denote the particle's 4-velocity by u,

$$u = u^{\hat{\mu}} e_{\hat{\mu}} = u^{\hat{0}} e_{\hat{0}} + u^{\hat{i}} e_{\hat{i}} = \gamma e_{\hat{0}} + \gamma v^{\hat{i}} e_{\hat{i}} \tag{43}$$

then the velocities $v^{\hat{i}}$ that the observer u_{ob} measures are

$$v^{\hat{i}} = \frac{u^{\hat{i}}}{\gamma} = \frac{u^{\hat{i}}}{u^{\hat{0}}} = \frac{u \cdot e_{\hat{i}}}{-u \cdot e_{\hat{0}}} = \frac{u \cdot e_{\hat{i}}}{-u \cdot u_{ob}} \tag{44}$$

$$v^{\hat{i}} = -\frac{u \cdot e_{\hat{i}}}{u \cdot u_{ob}} \tag{45}$$

Measured velocities

Lorentz transformations relate the frames of any two observers at the same point in spacetime, since they transform orthonormal basis to another orthonormal basis.

2.1 Observers in Schwarzschild Geometry

2.1.1 Stationary Observer

Two particles fall radially in from infinity in the Schwarzschild geometry. One starts with $e = 1$ and the other with $e = 2$. A stationary observer at radius $r = 6M$ measures the speed of each when they pass by. How much faster is the second particle moving at that point?

The Schwarzschild metric is

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega^2 \tag{46}$$

where

$$f(r) = 1 - \frac{2M}{r} \tag{47}$$

The timelike orthonormal basis vector is, by (16),

$$(e_{\hat{t}})^\mu = \left(f^{-\frac{1}{2}}, 0, 0, 0\right) \tag{48}$$

This is the observer's 4-velocity u_{ob}

$$(u_{ob})^\mu = (e_{\hat{t}})^\mu = \left(f^{-\frac{1}{2}}, 0, 0, 0\right) \tag{49}$$

The conserved quantity e is

$$e = -\xi \cdot u_p = -g_{tt} u_p^t = f u_p^t \tag{50}$$

where ξ is the timelike Killing vector and u_p is the particle's 4-velocity.

The measured energy per unit mass is

$$\frac{E}{m} = -u_{ob} \cdot u_p = -g_{tt} u_{ob}^t u_p^t = -\left(-f\right) f^{-\frac{1}{2}} e f^{-1} = e f^{-\frac{1}{2}} \tag{51}$$

Also

$$\frac{E}{m} = \gamma_{v_p} = \left(1 - \boldsymbol{v}_p^2\right)^{-\frac{1}{2}} \tag{52}$$

where $\boldsymbol{v}_p$ is the measured 3-velocity of the particle. We get the measured speed

$$v_p^2 = 1 - e^{-2} f \tag{53}$$

$$v_p = \frac{1}{e}\left(e^2 - 1 + \frac{2M}{r}\right)^{\frac{1}{2}} \tag{54}$$

For $r = 6M$

$$v_p = \frac{1}{e}\left(e^2 - \frac{2}{3}\right)^{\frac{1}{2}} \tag{55}$$

the speeds ratio is

$$\frac{v_p\left(e=2\right)}{v_p\left(e=1\right)} = \frac{\left(1/2\right)\left(10/3\right)^{1/2}}{\left(1/3\right)^{1/2}} = \frac{1}{2}\sqrt{10} \approx 1.58 \tag{56}$$

Homework exercise: Calculate (54) using (45).

2.1.2 Moving Observer

The stationary observer observes a second observer moving radially inward with speed v_r, at the same time and place that the particle passes by. What speed does the moving observer measure for the particle?

For pedagogical reason, we will do the long way, so we will see some relations. We denote the 4-vector of the moving observer by w. As before, we would like to compute

$$E'/m = -w \cdot u_p = \gamma_{v_p'} \tag{57}$$

where E' and $\gamma_{v_p'} = \left(1 - \boldsymbol{v}'^2\right)^{-\frac{1}{2}}$ are the energy and velocity of the particle that the moving observer measures.

The moving observer has a radial velocity, therefore his velocity 4-vector w

has the following form in coordinate basis

$$w^{\mu} = \left(w^t, w^r, 0, 0\right) \tag{58}$$

We wish to find w^t and w^r and plug into (57). The stationary observer u_{ob} measures the moving observer w to have a 3-velocity v_r, therefore

$$-u_{ob} \cdot w = \gamma_{v_r} \tag{59}$$

Plug (49) and (58) into (59)

$$-g_{tt} u^t_{ob} w^t = f f^{-\frac{1}{2}} w^t = \gamma_{v_r} \tag{60}$$

$\Rightarrow$

$$w^t = \gamma_{v_r} f^{-\frac{1}{2}} \tag{61}$$

Now we can find w^r by normalization constraint

$$w \cdot w = g_{tt} \left(w^t\right)^2 + g_{rr} \left(w^r\right)^2 = -1 \tag{62}$$

$$-f\left(\gamma_{v_r} f^{-\frac{1}{2}}\right)^2 + f^{-1} \left(w^r\right)^2 = -1 \tag{63}$$

$$\left(w^r\right)^2 = \left(\gamma^2_{v_r} - 1\right) f \tag{64}$$

$$w^r = \left(\gamma^2_{v_r} - 1\right)^{\frac{1}{2}} f^{\frac{1}{2}} \tag{65}$$

Notice that

$$\left(\gamma^2_{v_r} - 1\right)^{\frac{1}{2}} = \left(\frac{1}{1 - \boldsymbol{v}^2_r} - 1\right)^{\frac{1}{2}} = \left(\frac{\boldsymbol{v}^2_r}{1 - \boldsymbol{v}^2_r}\right)^{\frac{1}{2}} = \pm v_r \gamma_{v_r} \tag{66}$$

we take the negative square root since the moving observer is moving radially inward. We conclude that

$$w^r = -v_r \gamma_{v_r} f^{\frac{1}{2}} \tag{67}$$

Collecting (61) and (67)

$$\boxed{w^{\mu} = \left(\gamma_{v_r} f^{-\frac{1}{2}}, -v_r \gamma_{v_r} f^{\frac{1}{2}}, 0, 0\right)} \tag{68}$$

The moving observer 4-velocity in coordinate basis

Notice that the orthonormal basis vector $e_{\hat{t}}$ in the coordinate basis is given by (48) and $e_{\hat{r}} = \frac{e_r}{\sqrt{g_{rr}}}$ is

$$(e_{\hat{r}})^\mu = \left(0, f^{\frac{1}{2}}, 0, 0\right) \tag{69}$$

so

$$w = \gamma_{v_r} f^{-\frac{1}{2}} e_t - v_r \gamma_{v_r} f^{\frac{1}{2}} e_r \tag{70}$$

$$w = \gamma_{v_r} e_{\hat{t}} - v_r \gamma_{v_r} e_{\hat{r}} \tag{71}$$

The moving observer 4-velocity in orthonormal basis is by Lorentz trans.

w in orthonormal basis (71) is related to $u_{ob} = e_{\hat{t}}$ by a Lorentz transformation of boost with velocity $v_r e_{\hat{r}}$. But of course, that is what Lorentz transformations do! They relate different observer's frames at a point in spacetime.

We continue the calculation of (57) the hard way (in coordinate basis).

$$-w \cdot u_p = -g_{tt} w^t u_p^t - g_{rr} w^r u_p^r = \gamma_{v_p'} \tag{72}$$

It still remains to find u_p^r. We do it by normalization of u_p

$$u_p \cdot u_p = g_{tt} \left(u_p^t\right)^2 + g_{rr} \left(u_p^r\right)^2 = -1 \tag{73}$$

Plug (50) and the metric

$$-f \left(e f^{-1}\right)^2 + f^{-1} \left(u_p^r\right)^2 = -1 \tag{74}$$

from (54)

$$u_p^r = -\sqrt{e^2 - f} = -e v_p \tag{75}$$

Now plug (61),(67),(75) and the metric in (72)

$$f \left(\gamma_{v_r} f^{-\frac{1}{2}}\right) \left(e f^{-1}\right) - f^{-1} \left(-v_r \gamma_{v_r} f^{\frac{1}{2}}\right) \left(-e v_p\right) = \gamma_{v_p'} \tag{76}$$

$$\gamma_{v_r} f^{-\frac{1}{2}} e - v_r v_p \gamma_{v_r} f^{-\frac{1}{2}} e = \gamma_{v_p'} \tag{77}$$

Recall from (51) and (52) that $e f^{-\frac{1}{2}} = \gamma_{v_p}$, hence

$$\gamma_{v_p'} = \gamma_{v_r} \gamma_{v_p} \left(1 - v_r v_p\right) \tag{78}$$

v_p' can be extracted from (78). v_r is given and v_p we found in (54). Homework exercise: derive (78) by calculating (57) in orthonormal basis.

Let us show that (78) is simply the relation of the gamma's analogue to the addition of velocities law arising from Lorentz transformation. By hyperbolic functions identities, the addition of velocities formula is

$$v_p = tanh\,(\phi) \tag{79}$$

$$v_r = tanh\,(\psi) \tag{80}$$

$$v'_p = tanh\,(\psi - \phi) = \frac{v_r - v_p}{1 - v_r v_p} \tag{81}$$

The corresponding relation for the gamma's is

$$\gamma_{v_p} = cosh\,(\phi) \tag{82}$$

$$\gamma_{v_r} = cosh\,(\psi) \tag{83}$$

$$\begin{aligned} \gamma_{v'_p} &= cosh\,(\psi - \phi) = cosh\,(\psi)\,cosh\,(\phi) - sinh\,(\psi)\,sinh\,(\phi) \\ &= cosh\,(\psi)\,cosh\,(\phi)\,(1 - tanh\,(\psi)\,tanh\,(\phi)) = \gamma_{v_r}\gamma_{v_p}\,(1 - v_r v_p) \end{aligned} \tag{84}$$

To conclude, the velocity that the moving observer measures is related to the velocity that the stationary observer measures by the addition of velocities formula

$$v'_p = \frac{v_r - v_p}{1 - v_r v_p} \tag{85}$$

The short way is, after all, to use the Lorentz transformations that relate what different observers with relative velocity measure at the same event.