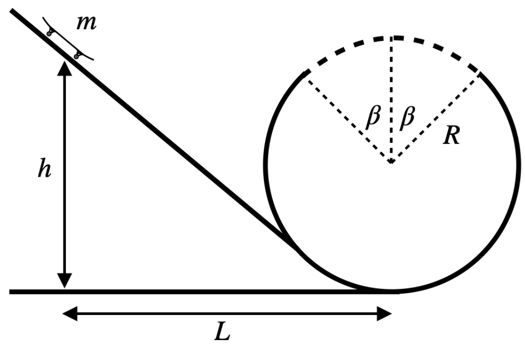
HW 7

1 Loop The Loop

A skate with mass m is sliding on a frictionless track which begins with a slope and ends with a loop with radius R, as shown in the figure.



1. What is the minimal height h, for which the skate will not leave the track at any point?
2. Due to faulty construction, the top part of the track (corresponds to angle 2β) collapsed. What should be the value of h so that the skate would complete a full loop on the track? Which value of β corresponds to minimal height h?
3. Consider the section of motion between $0 < x < L$ (where L is the distance to the bottom of the loop) and show that the velocity of the skate in the x direction follows

$$\dot{x} = \sqrt{\frac{2g\left(h - y\left(x\right)\right)}{1 + \left(\frac{dy}{dx}\right)^2}}.$$

Solution:

1. For the skate to stay on the track we require that the normal force between the skate and the track will be $N \geq 0$ at all times, where equality denotes the limit in which the skate parts from the track. Thus we need to look into the limit scenario in order to find the minimal height. Since the force which acts to pull the skate from the track is the gravitational force, the point in which its y component is largest is at the top of the loop.
Writing the equation of motion for the skate, at the top of the loop we find

$$N - mg = ma_r \quad \overset{N=0}{\longrightarrow} \quad v^2 = gR.$$

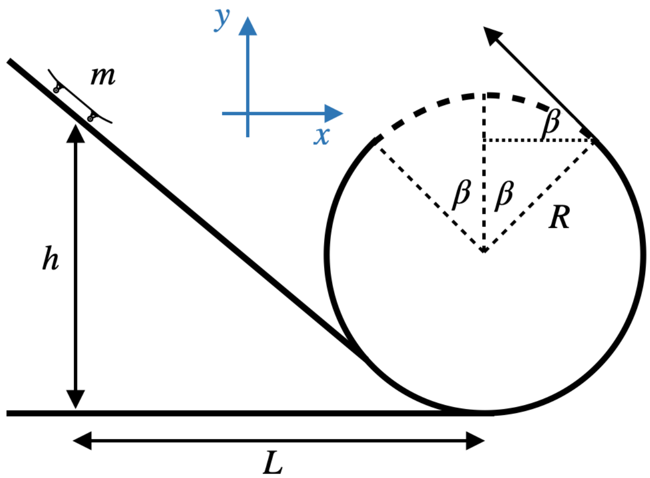
In order to find the relation between h and v^2 we employ conservation of energy

$$mgh = mg\left(2R\right) + \frac{1}{2}mv^2 \quad \rightarrow \quad h = \frac{v^2}{2g} + 2R,$$

plugging v^2 yields

$$h = \frac{5R}{2}.$$

2. Throughout the motion along the collapsed part of the track the skate undergoes free-fall motion. Therefore, we need to require that the skate's displacement in the horizontal direction is $2R\sin\beta$ as it reaches the same height it was when it left the track.



The initial velocity of the skate when it leaves the track is found from energy conservation

$$mgh = mgR(1+\cos\beta) + \frac{1}{2}mv_0^2 \quad \rightarrow \quad v_0^2 = 2g\left[h - R(1+\cos\beta)\right].$$

The time it takes for the skate to return to the same height is found by considering kinematics vertical direction:

$$0 = v_0 \sin\beta\, t - \frac{g}{2}t^2 \quad \rightarrow \quad t_f = 2\frac{v_0\sin\beta}{g},$$

thus, the horizontal displacement is

$$|x| = v_0\cos\beta\left(2\frac{v_0\sin\beta}{g}\right) = 2\frac{v_0^2}{g}\cos\beta\sin\beta \overset{!}{=} 2R\sin\beta \quad \rightarrow \quad \frac{v_0^2}{g}\cos\beta = R.$$

Plugging in the expression for v_0^2 we find

$$2\left[h - R(1+\cos\beta)\right]\cos\beta = R \quad \rightarrow \quad h = R\left(\frac{1}{2\cos\beta} + 1 + \cos\beta\right).$$

In order to find the minimal value of h we take the derivative and find β which minimizes h

$$\frac{dh}{d\beta} = R\left(\frac{\sin\beta}{2\cos^2\theta} - \sin\beta\right) = 0 \quad \rightarrow \quad \cos\beta = \frac{1}{\sqrt{2}} \quad \rightarrow \quad \beta = \frac{\pi}{4}.$$

This result is reasonable since we know that 45° throw yields maximum distance for certain velocity, which can be reversed to minimal velocity for certain distance.

3. Let us write down the conservation of energy for any point on the slope

$$mgh = mgy + \frac{1}{2}mv^2,$$

but $v^2 = \dot{x}^2 + \dot{y}^2$, thus

$$\dot{x}^2 + \dot{y}^2 = 2g(h-y).$$

Since we need to find $\dot{x}(x)$ let us convert the time derivative of y to

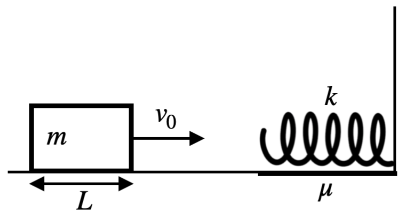
$$\dot{y} = \frac{dy}{dx}\dot{x},$$

hence

$$\dot{x} = \sqrt{\frac{2g(h - y(x))}{1 + \left(\frac{dy}{dx}\right)^2}}.$$

2 Mass, Spring and Friction

A body with length L, with a uniform mass distribution and mass m is moving on a frictionless surface with velocity v_0. The body then reaches a region of the surface with friction coefficient $\mu = \mu_k = \mu_s$, on which lies a massless spring with coefficient k, which is connected to a wall, as shown in the figure.



1. What is the value of v_0, for which the body will stop exactly when its entire length is on the surface with the friction?
2. Given v_0 is the value from (1), what is the maximal value of μ that allows the complete return of the body?
3. Given v_0 is the value from (1) and μ from (2), what would be the body's final velocity?

Solution:

1. The equation for conservation of energy yields

$$\underbrace{\frac{1}{2}mv_0^2}_{E_k} = \underbrace{\frac{1}{2}kL^2}_{U_e} + \underbrace{|W_f|}_{\text{energy loss}} .$$

The work done by the friction force is

$$W_f = \int_0^L \mu N dx,$$

where the normal force is x dependent $N(x) = \frac{x}{L}mg$. Thus

$$W_f = \mu\frac{mg}{L}\int_0^L x\,dx = \frac{\mu mgL}{2} \quad \rightarrow \quad v_0 = \sqrt{\frac{k}{m}L^2 + \mu gL}.$$

2. Writing the expressions for both forces: elastic and friction for the interval $0 < x < L$, we find that

$$F_e = -kx$$
$$f = \mu\frac{x}{L}mg.$$

In order to have movement to the left we must require

$$F_e \geq f \quad \rightarrow \quad \mu \geq \frac{Lk}{mg}.$$

3. Again, using energy conservation

$$\frac{1}{2}kL^2 = \frac{1}{2}mv_f^2 + |W_f| \quad \rightarrow \quad v_f = \sqrt{\frac{k}{m}L^2 - \mu gL}.$$

3 Work of a changing force

A changing force acting on a mass of $5kg$ with an initial velocity of $4\frac{m}{sec}$.
The force depends on the displacement x and is different for different values of x according to:

$$\begin{array}{lc} F_1(x) = 2x & 0 < x < 5 \\ F_2(x) = 10 & 5 < x < 15 \\ F_3(x) = 20 - 2x & 15 < x < 20 \end{array}$$

a. What is the work of this force along each different section of x?

b. What will be the change in kinetic energy after the mass passed $20\,meters$? Assume this is the only force on the mass.

c. What is the velocity of the mass after $20\,meters$?

Solution:

a.

$$W_1 = \int_0^5 2x\,dx = \left[x^2\right]_0^5 = 25\,joule$$

$$W_2 = \int_5^{15} 10\,dx = [10x]_5^{15} = 100\,joule$$

$$W_3 = \int_{15}^{20} 40 - 2x\,dx = \left[40x - x^2\right]_{15}^{20} = 25\,joule$$

b. According to work - energy theorem the change in kinetic energy is given by

$$\Delta K = W_{tot} = W_1 + W_2 + W_3 = 150\,joule$$

c.

$$\Delta K = \frac{1}{2}m\left(v_f^2 - v_0^2\right)$$

$$v_f^2 = v_0^2 + \frac{2\Delta K}{m} = 16\frac{m^2}{sec^2} + \frac{2\cdot 150}{5}\frac{joule}{kg}$$

Note that $\frac{joule}{kg} = \frac{m^2}{sec^2}$, and we get

$$v_f = 8.7\frac{m}{sec}.$$

4 Conservative Force

For the given force

$$\boldsymbol{F} = -k\,(x-y)^2\,\hat{\boldsymbol{x}} + k\,(x-y)^2\,\hat{\boldsymbol{y}} + Kz\hat{\boldsymbol{z}},$$

1. Is the force a conservative force? If it is not, demonstrate it. If it is, find its potential.
2. A body moves from position $(0,0,0)$ in a straight line to position $(D,D,0)$, and then to (D,D,D). Calculate the work done by the force, using path integral.

Solution:

1. Let us calculate the rotor of $\boldsymbol{F}$

$$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{F} &= (\partial_y F_z - \partial_z F_y)\,\hat{\boldsymbol{x}} + (\partial_z F_x - \partial_x F_z)\,\hat{\boldsymbol{y}} + (\partial_x F_y - \partial_y F_x)\,\hat{\boldsymbol{z}} \\ &= (0-0)\,\hat{\boldsymbol{x}} + (0-0)\,\hat{\boldsymbol{y}} + [2k\,(x-y) - 2k\,(x-y)]\,\hat{\boldsymbol{z}} \\ &= 0.\end{aligned}$$

The force is conservative, let us find its potential U, which must follow $\boldsymbol{F} = -\boldsymbol{\nabla}U$

$$\begin{aligned}U &= -\int F_x dx = \frac{k}{3}\,(x-y)^3 + C_x\,(y,z)\,, \\ U &= -\int F_y dy = \frac{k}{3}\,(x-y)^3 + C_y\,(x,z)\,, \\ U &= -\int F_z dz = -\frac{K}{2}z^2 + C_z\,(x,y)\,.\end{aligned}$$

Thus

$$\frac{k}{3}\,(x-y)^3 + C_x\,(y,z) = \frac{k}{3}\,(x-y)^3 + C_y\,(x,z) = -\frac{K}{2}z^2 + C_z\,(x,y)\,,$$

from the left equation we conclude $C_x = C_y = C\,(z)$, then from the right equation we find

$$\begin{aligned}C\,(z) &= -\frac{K}{2}z^2 + C_0 \\ C_z\,(x,y) &= \frac{k}{3}\,(x-y)^3 + C_0.\end{aligned}$$

Plugging all into U we find

$$U = \frac{k}{3}\,(x-y)^3 - \frac{K}{2}z^2 + \cancel{C_0}.$$

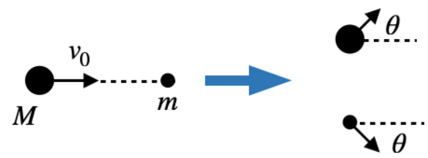
2. The path integral is

$$\begin{aligned}W &= \int \boldsymbol{F}\cdot d\boldsymbol{r} \\ &= \int_{(0,0,0)}^{(D,D,0)} \boldsymbol{F}\cdot d\boldsymbol{r} + \int_{(D,D,0)}^{(D,D,D)} \boldsymbol{F}\cdot d\boldsymbol{r} \\ &= \int_{(0,0)}^{(D,D)} \left[-k\,(x-y)^2\,\hat{\boldsymbol{x}} + k\,(x-y)^2\,\hat{\boldsymbol{y}}\right]\cdot[d\boldsymbol{x} + d\boldsymbol{y}] - \int_0^D Kz\hat{\boldsymbol{z}}\cdot d\boldsymbol{z} \\ &= \int_0^D -k\,(x-y\,(x))^2\,dx + \int_0^D k\,(x\,(y)-y)^2\,dy - \int_0^D Kz\,dz \\ &= \int_0^D -k\cancel{(x-x)}^2 dx + \int_0^D k\cancel{(y-y)}^2 dy + \frac{1}{2}KD^2 \\ &= \frac{1}{2}KD^2,\end{aligned}$$

where the first integral is along the line $y = x$, and since $x - y = 0$ it vanishes along this path.

5 Scattering

A body with mass $M = \alpha m$ moves with velocity v_0 towards a stationary body with mass m. After the collision between the two bodies, each body moves at opposite direction with the same angle relative to the original direction of motion (see figure).
Notice: the velocities are not necessarily equal.
Given that the collision between the bodies was elastic (energy was conserved), and that the process happens on a horizontal plane,



1. Assume that $\alpha = 2$. What is θ?
2. What is the maximum possible ration $\alpha = M/m$ for such scattering?

Solution:

1. Since all the forces act on the bodies are internal (within the system of the two bodies), the total momentum is conserved. Therefore

$$\boldsymbol{p}_i = (\alpha m v_0, 0)$$
$$\boldsymbol{p}_f = (P_x + p_x, P_y + p_y)\,,$$

thus

$$\alpha m v_0 = P_x + p_x \quad P_y = -p_y.$$

Since the directions are opposite, we have

$$\left|\frac{P_y}{P_x}\right| = \left|\frac{p_y}{p_x}\right| \quad \rightarrow \quad P_x = p_x,$$

$$P_x = p_x = \frac{1}{2}\alpha m v_0.$$

In order to find the y direction quantities we use energy conservation,

$$\frac{1}{2}\alpha m v_0^2 = \frac{P_x^2 + P_y^2}{2\alpha m} + \frac{p_x^2 + p_y^2}{2m}$$

$$P_y^2 = \frac{\alpha^2 m^2 v_0^2}{4}\left(\frac{3-\alpha}{1+\alpha}\right) \quad \rightarrow \quad P_y = \pm\frac{\alpha m v_0}{2}\sqrt{\frac{3-\alpha}{1+\alpha}} \quad \rightarrow \quad \tan\theta = \frac{P_y}{P_x} = \pm\sqrt{\frac{3-\alpha}{1+\alpha}}.$$

Then, for $\alpha = 2$,

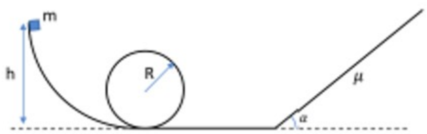
$$\tan\theta = \pm\sqrt{\frac{1}{3}} \quad \rightarrow \quad \theta = \pm\frac{\pi}{6}.$$

2. In order to find the maximum possible ration we require $\alpha < 3$, thus

$$0 < \frac{M}{m} < 3.$$

6 Loop of death - Bonus

A track consists of a smooth track, a loop of death of radius R, and a long rough slope with a friction coefficient μ and an angle α (as shown in the figure).
From what minimal height h we need to release the body in the left track in order to it pass the loop of death twice?

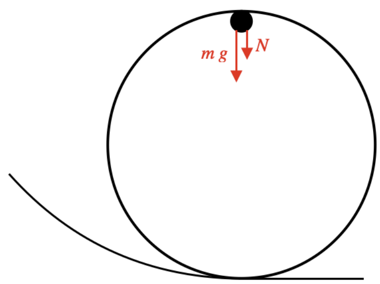


Solution:

Let's write the total energy of the body

$$E = K + V = \frac{1}{2}mv^2 + mgh$$

Starting from the end, the body will preform a circular motion on the loop of death for the second time if at the upper point of the loop the body will still touch it. Meaning that $N > 0$.



Writing the force equation:

$$N + mg = m\frac{v^2}{R}$$

$$0 < N = m\left(\frac{v^2}{R} - g\right)$$

$$v_{min}^2 = gR$$

Therefore

$$(E_2)_{MIN} = \frac{1}{2}m\,(gR) + mg2R = \frac{5}{2}mgR$$

where E_2 is the total energy of the body on the top point of the loop of death when it passes it for the second time.
Denote E_1 for the total energy of the body when it stops at some height marked as H in the rough slope.
The difference between those two is equal to the work done by friction.
The normal force when the body is on the rough slope

$$N = mg\cos\alpha$$

therefore the friction force is equal to $|f| = \mu mg\cos\alpha$ and directed up the slope.
Calculate the friction work as the body slides down a distance d

$$H = d\sin\alpha$$

and the friction is constant along the slide

$$W_f = -\mu mgd\cos\alpha.$$

Now we can get the minimal value for E_1:

$$(E_2)_{MIN} = (E_1)_{MIN} + W_f$$

$$(E_1)_{MIN} = \frac{5}{2}mgR + \mu mgd\cos\alpha.$$

We can find d by writting explicitly $(E_1)_{MIN}$:

$$(E_1)_{MIN} = mgH = mgd\sin\alpha$$

and

$$\frac{5}{2}R = d\left(\sin\alpha - \mu\cos\alpha\right).$$

Denote E_0 for the total energy of the body when it first released at the left track.
$E_1 - E_0$ equal to the work done by friction when the body slided up the slope. Note that it is the same trajectory as before in the opposite direction, with a friction force acting to an opposite direction as well. Therefore

$$mgh = (E_0)_{MIN} = \frac{5}{2}mgR + 2\mu mgd\cos\alpha.$$

$$h = \frac{5}{2}R\left(1 + \frac{2\mu\cos\alpha}{\sin\alpha - \mu\cos\alpha}\right) = \frac{5}{2}R\left(\frac{\sin\alpha + \mu\cos\alpha}{\sin\alpha - \mu\cos\alpha}\right).$$

7 Potential Energy - Bonus

Force is given by $F_\rho = \frac{a}{\rho^2}$, $F_\varphi = \frac{b\sin\varphi}{\rho^2}$ (cylindrical coordinates).
Is the force conservative?
*Note that in cylindrical coordinates the rotor is given by:

$$\vec{\nabla} \times \vec{F} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\varphi} & \hat{z} \\ \partial_\rho & \partial_\varphi & \partial_z \\ F_\rho & \rho F_\varphi & F_z \end{vmatrix}$$

If yes, find the potential energy.
What is conserved in this force?

Solution:

The force is conservative if

$$\vec{\nabla} \times \vec{F} = 0$$

$$\vec{\nabla} \times \vec{F} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\varphi} & \hat{z} \\ \partial_\rho & \partial_\varphi & \partial_z \\ \frac{a}{\rho^2} & \rho\left(\frac{b\sin\varphi}{\rho^2}\right) & 0 \end{vmatrix} =$$

$$= \frac{\left[-\partial_z\left(\rho\left(\frac{b\sin\varphi}{\rho^2}\right)\right)\right]\hat{\rho} + \left[\partial_z\left(\frac{a}{\rho^2}\right)\right]\hat{\varphi} + \left[\partial_\rho\left(\rho\left(\frac{b\sin\varphi}{\rho^2}\right)\right) - \partial_\varphi\left(\frac{a}{\rho^2}\right)\right]\hat{z}}{\rho} \Rightarrow$$

where $\overrightarrow{F}$ is not dependent on z and F_ρ is not dependent on φ, so

$$\Rightarrow \frac{\partial_\rho\left(\frac{b\sin\varphi}{\rho}\right)}{\rho}\hat{z} = \frac{-\frac{b\sin\varphi}{\rho^2}}{\rho}\hat{z} \Rightarrow \overrightarrow{\nabla} \times \overrightarrow{F} = -\frac{b\sin\varphi}{\rho^3} \neq 0$$

The force is not conservative for $b \neq 0$.
We could equally check if mixed partial are equal:

$$\frac{1}{\rho}\frac{\partial F_\rho}{\partial\varphi} = -\frac{1}{\rho}\frac{\partial^2 V}{\partial\varphi\partial\rho} = 0$$

$$\frac{\partial F_\varphi}{\partial\rho} = -\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial V}{\partial\varphi}\right) = \frac{b\sin\varphi}{\rho^2}$$

$$\frac{\partial F_\varphi}{\partial\rho} \neq \frac{1}{\rho}\frac{\partial F_\rho}{\partial\varphi}$$

Which means that there is no potential function V for $\overrightarrow{F}$.