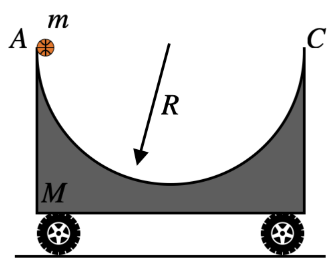
HW 8

1 Sliding Cart

A cart with mass M stationed on a frictionless horizontal plane. On the cart installed a frictionless surface in the shape of half a circle with radius R (see figure). A ball with mass m is allowed to slide from point A (the top of the plane).



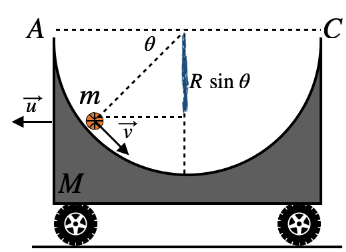
1. Write down the equations for momentum and energy conservation. Denote the ball's velocity components (relative to stationary frame) by v_x and v_y, and those of the cart by u_x and u_y.
2. Does the ball reach the other side of the surface (point C)? If so, find its velocity and the position of the cart (relative to stationary frame) at the moment the ball reaches point C.

Solution:

1. Let us define the point of reference for the potential energy to be at the height of point A (or C equivalently), thus at any point

$$E_p + E_k = -mgR\sin\theta + \frac{1}{2}m\left(v_x^2 + v_y^2\right) + \frac{1}{2}Mu_x^2 = \text{Constant},$$

where we defined θ as shown in the figure



The velocity of the ball, relative to the cart, is

$$\boldsymbol{v}' = \boldsymbol{v} - \boldsymbol{u} = \boldsymbol{v} - u_x\hat{\boldsymbol{x}},$$

therefore

$$\sin\theta = \frac{|v_x'|}{|\boldsymbol{v}'|} = \frac{|v_x - u_x|}{|\boldsymbol{v} - u_x\hat{\boldsymbol{x}}|} = \frac{|v_x - u_x|}{\sqrt{(v_x - u_x)^2 + v_y^2}}.$$

Plugging that into the energy conservation equation we find

$$-mgR\frac{|v_x - u_x|}{\sqrt{(v_x - u_x)^2 + v_y^2}} + \frac{1}{2}m\left(v_x^2 + v_y^2\right) + \frac{1}{2}Mu_x^2 = 0.$$

As for the momentum conservation, the momentum is only conserved in the x direction, in which

$$mv_x + Mu_x = 0.$$

2. At point C the ball would have $v_y = 0$, and since the ball does not leave the surface, we must also require that at this point $v_x = u_x$. Looking at the momentum conservation we find that the only configuration for this to happen is when

$$u_x = v_x = 0,$$

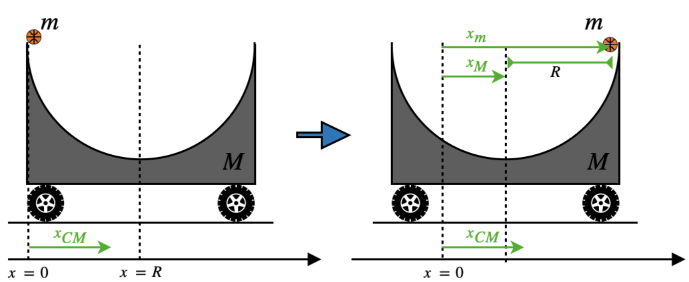
that means that the ball reaches point C where its velocity becomes 0. (we could also see that from the expression for $\sin\theta$ at $\theta = \pi$)
Since no external forces act on the x direction, we can expect the center of mass to remain constant, defining the origin at point A, we find the initial position of the CM to be

$$x_{CM}(t_A) = \frac{mx_m(t_A) + Mx_M(t_A)}{m+M} = \frac{0 + MR}{m+M} = \frac{MR}{m+M}.$$

Therefore, when the ball reaches point C we find

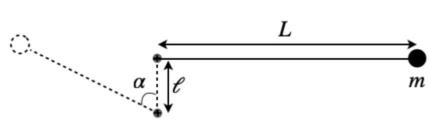
$$x_{CM}(t_C) = \frac{mx_m(t_C) + Mx_M(t_C)}{m+M} = \frac{m(x_M(t_C) + R) + Mx_M(t_C)}{m+M} = \frac{MR}{m+M} \quad \rightarrow \quad x_M(t_C) = R\frac{M-m}{m+M},$$

implying that it really did move left.



2 Yo-Yo

A yo-yo with mass m is connected to a nail via string with length L. Underneath the nail, at distance ℓ, another nail is placed (see figure). The yo-yo is released from rest, and when it reaches the second nail it continues motion as the string remains stretched between the two nails.



1. What is the velocity of the yo-yo when the angle between the two segments of the string reaches α (see figure)?

2. Find α for which the string becomes loose.

Solution:

1. Since the tension acts perpendicular to the path of the yo-yo, and the only other force is gravity, which is conservative, then the energy is conserved during the motion. Choosing the point of reference for the potential energy to be at the bottom nail we can write

$$E_i = mg\ell$$
$$E_f = \frac{1}{2}mv_\alpha^2 + mg\left(L-\ell\right)\cos\alpha,$$

which gives us

$$v_\alpha = \sqrt{2g\left[\ell\left(1+\cos\alpha\right) - L\cos\alpha\right]}.$$

2. The yo-yo undergoes circular motion around the bottom nail up to the point when the string becomes loose. Therefore we may write the force equation as

$$-T - mg\cos\alpha = -m\frac{v^2}{L-\ell} = -m\frac{2g}{L-\ell}\left[\ell\left(1+\cos\alpha\right) - L\cos\alpha\right].$$

At the point of loosening the tension vanishes, therefore

$$\cos\alpha = \frac{2}{L-\ell}\left[\ell + \left(\ell - L\right)\cos\alpha\right] \quad \rightarrow \quad \cos\alpha = \frac{2}{3}\frac{\ell}{L-\ell}.$$

We see that when $\ell \to L$ the solution breaks down since the yo-yo does not reach the bottom nail, and also there is a minimal ℓ when we require $\cos\alpha \le 1$, we find $\ell \ge \frac{3}{5}L$, over which the yo-yo will complete a loop (i.e. the string will never be loose).

3 Central potential and Ionization

A particle is connected to a point (The source of some force that the particle feels) by a central potential

$$u(r) = \frac{A}{r^2} - \frac{B}{r}$$

where r is the distance of the particle from the source of the force and A and B are positive constants.
* The motion of the particle in the radial direction only and therefore can be described as one-dimensional.

1. Find the distance of equilibrium r_0 and show that it is stable.
2. Calculate the energy of ionization ϵ_0 i.e. the amount of work needed in order to move the particle from $r_0$to infinity.
3. Express $u(r)$ in terms of r_0 and ϵ_0 alone.
4. What is the work done by the force derived from $u(r)$ when the particle moving from $r_1 = \left(\sqrt{2}r_0, \sqrt{2}r_0\right)$ to $r_2 = \left(\frac{r_0}{\sqrt{2}}, \frac{r_0}{\sqrt{2}}\right)$ in the $x-y$ plane? Use ϵ_0 to express your answer.
5. It is given that the total energy of the particle is $E = -\frac{3\epsilon_0}{4}$ and that its motion is only radial. Find the values of r where the particle's velocity vanishes.

Solution:

1. Differentiating $u(r)$

$$\frac{du}{dr} = -2\frac{A}{r^3} + \frac{B}{r^2}$$

$\frac{du}{dr} = 0$ for the distance of equilibrium r_0

$$2\frac{A}{r_0^3} = \frac{B}{r_0^2}$$

$$r_0 = \frac{2A}{B}.$$

If $r_0$is a stable equilibrium $\left(\frac{d^2u}{dr^2}\right)_{r_0} > 0$

$$\left(\frac{d^2u}{dr^2}\right)_{r_0} = \left(6\frac{A}{r^4} - 2\frac{B}{r^3}\right)_{r_0} =$$

$$= 6A\left(\frac{B}{2A}\right)^4 - 2B\left(\frac{B}{2A}\right)^3 =$$

$$= \frac{3}{8}\frac{B^4}{A^3} - \frac{2}{8}\frac{B^4}{A^3} > 0$$

2. Using the relation between potential energy and work by its force:

$$\epsilon_0 \equiv W(r_0 \to \infty) = \Delta u = u(\infty) - u(r_0) = -u(r_0) = \frac{B^2}{4A}$$

*See the note at the end of this solution.

3. $u(r) = \epsilon_0\left[\left(\frac{r_0}{r}\right)^2 - 2\left(\frac{r_0}{r}\right)\right]$

4. We note that this force is conservative (it has a given potential) therefore the trajectory does not important - only the initial and final points:

$$\begin{aligned} W &= u(r_2) - u(r_1) = \\ &= u\left(\sqrt{\frac{r_0^2}{2} + \frac{r_0^2}{2}}\right) - u\left(\sqrt{2r_0^2 + 2r_0^2}\right) = u(r_0) - u(2r_0) = \\ &= -\epsilon_0 - \left(\epsilon_0\left[\left(\frac{r_0}{2r_0}\right)^2 - 2\left(\frac{r_0}{2r_0}\right)\right]\right) = -\epsilon_0 + \frac{3}{4}\epsilon_0 = \\ &= -\frac{\epsilon_0}{4} \end{aligned}$$

*See the note at the end of this solution.

5. The total energy of the particle is given by

$$E_{tot} = K + u(r)$$

When the velocity of the particle vanishes its kinetic energy also vanishes.

$$\left(\frac{r_0}{r}\right)^2 - 2\left(\frac{r_0}{r}\right) = -\frac{3}{4}$$

$$r^2 - \left(\frac{8r_0}{3}\right)r + \frac{4}{3}r_0^2 = 0$$

$$r_{1/2} = \frac{1}{2}\left[\frac{8r_0}{3} \pm \sqrt{\left(\frac{8r_0}{3}\right)^2 - 4\frac{4}{3}r_0^2}\right] = \frac{4}{3}r_0\left[1 \pm \sqrt{1 - \frac{6}{8}}\right]$$

$$r_{1/2} = \frac{4}{3}r_0\left(1 \pm \frac{1}{2}\right)$$

*__Note:__ In section 2 and section 4 the particle moved in the same direction but we got opposite signs for the work.
In section 2 "we" needed to exert force on the particle **against the potential** in order to move it and that is why the work is positive.
In section 4 the potential did the work on the particle and that is why we got a negative work.

4 Work on a box

A box with a mass of $m = 150\,kg$ hangs from a rope of length $L = 15\,meters$.
A varying force $\boldsymbol{F}$push the box horizontally to move it a distance of $d = 4\,meters$ so that at the end of the movement the mass remains motionless as shown in the figure.

A. What is the magnitude of the force $\boldsymbol{F}$ at the end of the motion?

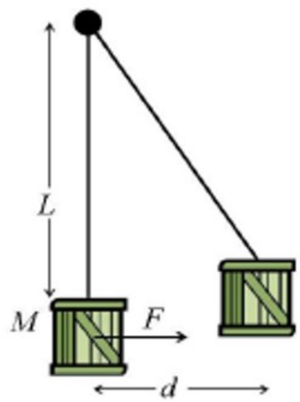
B. What is the total work done on the box during its movement?

C. What work is done on the box during its movement by gravity?

D. What is the work done on the box during its movement by the rope?

E. Given that before and after the motion the box remains in place. Use the answers to the previous sections to find the work done by the varying force $\boldsymbol{F}$.

F. Why is the work of $\boldsymbol{F}$ not equal to the product of the displacement in your answer to section **A**?



Solution:

A. The equation of motion when the box is it rest are:

$$\begin{array}{ll} \hat{x} & F - T\sin\theta = 0 \\ \hat{y} & T\cos\theta - mg = 0 \end{array}$$

The angle between the rope at the end and the vertical direction is given by

$$\sin\theta = \frac{d}{L} = \frac{4}{15}.$$

Then $T = \frac{mg}{\cos\theta}$ and

$$F = mg\tan\theta = 415\,N$$

B. The kinetic energy at the start and at the end of the motion is zero.
According to work - energy theorem

$$W_{tot} = \Delta K = 0$$

C. The location of the box is given by (origin at the upper end of the rope)

$$\boldsymbol{r} = (L\sin\theta, -L\cos\theta)$$

then the trajectory is given by

$$\boldsymbol{dl} = L(\cos\theta, \sin\theta)\, d\theta$$

for $\theta : 0 \to \theta$.

$$W_g = \int_0^\theta -mg\hat{y} \cdot L(\cos\theta, \sin\theta)\, d\theta = \int_0^\theta -mgL\sin\theta\, d\theta =$$

$$= [mgL\cos\theta]_0^\theta = mgL\left(\frac{L}{\sqrt{L^2+d^2}} - 1\right) = -815\, joule$$

D. The tension force acting on the mass by the rope is perpendicular to the trajectory during the whole motion.

$$W_{rope} = 0$$

E.

$$0 = W_{tot} = W_F + W_g + \cancel{W_{rope}}$$
$$W_F = -W_g = 815\, joule$$

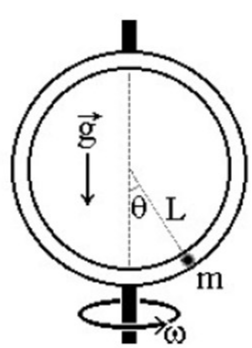
F.

$$F(\theta) \neq const$$
$$W_F = \int F dl \neq F \int dl.$$

5 Spherical pendulum

A bead of mass m is inside a hollowed hoop of radius L which is turning with an angular velocity ω around its axis (vertical relative to ground).

1. Write an expression for the total potential energy $V(\theta)$ of the bead in the frame rotating with the hoop.
2. Use the expression for $V(\theta)$ to find the angle where the bead is in equilibrium and check if they are stable.
3. Sketch $V(\theta)$.



Solution:

1. First let us recognize the forces in the rotating frame of reference:
   Gravitational force $F_g = mg(-\hat{y})$- which we know it is conservative.
   Centrifugal force $F_c = m\omega^2\rho(\theta)\hat{\rho}$ - A force in the $\hat{\rho}$ direction which only depend on ρ - its mixed partial must vanish - also a conservative force.
   The centrifugal potential

$$V_c = -\int m\omega^2 \rho d\rho = -\frac{1}{2}m\omega^2\rho^2 = -\frac{1}{2}m\omega^2 L^2 \sin^2\theta.$$

   The gravitational potential

$$V_g = mgL\left(1 - \cos\theta\right).$$

   The total potential energy

$$V(\theta) = mgL\left(1 - \cos\theta - \frac{1}{2}\frac{\omega^2 L}{g}\sin^2\theta\right).$$

2. The equilibrium point satisfies $\frac{dV}{d\theta} = 0$

$$mgL\left(\sin\theta - \frac{\omega^2 L}{g}\sin\theta\cos\theta\right)$$

$$\sin\theta = 0 \text{ or } \cos\theta = \frac{g}{\omega^2 L}$$

$$\theta_1 = 0, \pi \text{ or } \theta_2 = \pm\arccos\frac{g}{\omega^2 L}$$

   Differentiating again in order to find the stable points

$$\frac{d^2V}{d\theta^2} = \frac{d}{d\theta}\left(mgL\sin\theta\left(1 - \frac{\omega^2 L}{g}\cos\theta\right)\right) =$$
$$= mgL\left[\cos\theta\left(1 - \frac{\omega^2 L}{g}\cos\theta\right) + \frac{\omega^2 L}{g}\sin^2\theta\right]$$

   Define $\omega_c \equiv \sqrt{\frac{g}{l}}$

$$\left(\frac{d^2V}{d\theta^2}\right)_{\theta=0} = mgL\left(1 - \frac{\omega^2}{\omega_c^2}\right)$$

   For $\omega < \omega_c$ $\theta = 0$ is minimum and is stable. For $\omega < \omega_c$ $\theta = 0$ is unstable.

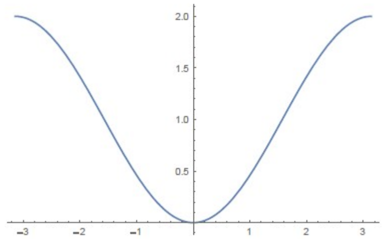
$$\left(\frac{d^2V}{d\theta^2}\right)_{\theta=\pi} = -mgL\left(1 + \frac{\omega^2}{\omega_c^2}\right)$$

   therefore $\theta = \pi$ is unstable for any value of ω.
   Note that for $\omega < \omega_c$ the points θ_2 are irrelevant.
   Assuming $\omega > \omega_c$**:**

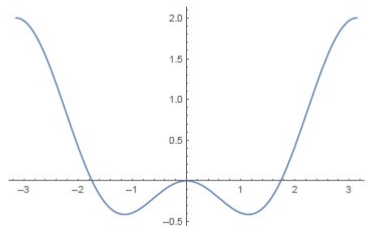
$$\left(\frac{d^2V}{d\theta^2}\right)_{\theta=\pm\arccos\frac{g}{\omega^2 L}} = mgL\left(\frac{g}{\omega^2 L}(1-1) + \frac{\omega^2 L}{g}\left(1 - \left(\frac{g}{\omega^2 L}\right)^2\right)\right) =$$
$$= mgL\left(\left(\frac{\omega}{\omega_c}\right)^2 + \left(\frac{\omega_c}{\omega}\right)^2\right) > 0$$

   and θ_2 are stable

3. The graph for $\omega < \omega_c$

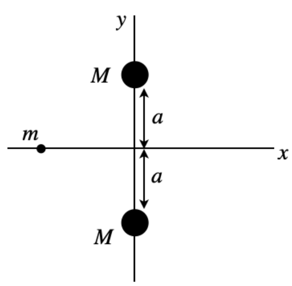


For $\omega > \omega_c$



6 Oscillations of bead with gravitating masses - Bonus

A bead of mass m slides without friction on a smooth rod along the x axis. The rod is equidistant between two spheres of mass M. The spheres are located at $x = 0$, $y = \pm a$ as shown in the figure, and attract the bead gravitationally.



Using the potential energy:

1. Find the point of equilibrium.
2. Find the frequency of small oscillations of the bead about the origin.
3. Draw the potential as a function of x.
4. For what range of energies do we find bounded trajectories?
5. Given energy $E < 0$, what would be the turning points of the trajectory?

Solution:

1. The point of equilibrium can be found directly from the potential which is

$$U = -2\frac{GmM}{\sqrt{a^2 + x^2}},$$

simply by finding its minima . Taking the derivative and equating it to zero yields

$$\frac{dU}{dx} = 2\frac{GmMx_0}{(a^2+x_0^2)^{3/2}} = 0 \quad \rightarrow \quad x_0 = 0.$$

2. The frequency of the oscillations may be extracted from the potential as well. Expanding the potential as a Taylor series around the point of equilibrium

$$U(x) = U(x_0) + \left.\frac{dU}{dx}\right|_{x_0}(x-x_0) + \frac{1}{2}\left.\frac{d^2U}{dx^2}\right|_{x_0}(x-x_0)^2 + \ldots$$

we find that the first derivative vanishes (as we required to be at x_0), while the second derivative reads

$$\frac{d^2U}{dx^2} = 2GmM\frac{a^2-2x^2}{(a^2+x^2)^{5/2}},$$

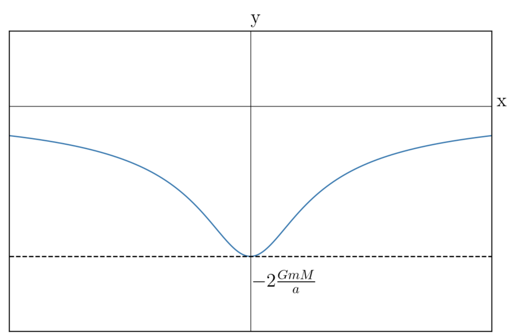
thus

$$U(x) = -2\frac{GmM}{\sqrt{a^2+0}} + GmM\frac{a^2-0}{(a^2+0)^{5/2}}(x-0)^2 + \cdots \approx -2\frac{GmM}{a} + \frac{1}{2}\left(\frac{GmM}{a^3}\right)x^2.$$

The first term is the minimal energy of the system, which is constant and does not affect physical quantities, while the second term is the leading term which exhibits the dynamics of the system. The latter resembles the potential energy of a spring, which is $\frac{1}{2}kx^2$, for which we already know the frequency of the oscillations to be $\omega = \sqrt{k/m}$, only with $k = GmM/a^3$. Therefore we can expect the frequency of the oscillations to be

$$\omega = \sqrt{\frac{GM}{a^3}}.$$

3. The potential is symmetric around the y axis, goes to zero for $x \rightarrow \pm\infty$ and has a minimum at $x = 0$ with the value $-2GmM/a$, thus



4. In order for the trajectory to be bounded we require that the trajectory will have turning points in which the kinetic energy vanishes. Due to energy conservation it is easy to see that

$$E = E_k + U \quad \overset{E_k=0}{\rightarrow} \quad E = U(x_{\max}) < 0 \quad \rightarrow \quad E < 0,$$

the energy must be negative, or in equivalently $|E_k(x)| < |U(x)|$. In other words we could say that the energy must be inside the potential well we drew in (3).

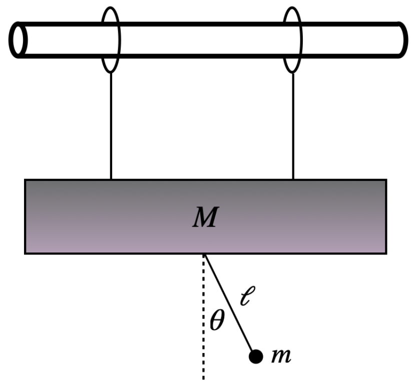
5. Given $E < 0$ we write again energy conservation and plug $E_k = 0$, which yields

$$-|E| = U(x_{\max}) = -2\frac{GmM}{\sqrt{a^2+x_{\max}^2}} \quad \rightarrow \quad x_{\max} = \sqrt{\left(\frac{2GmM}{|E|}\right)^2 - a^2},$$

which exhibits the property of increasing $x_{\max}$ as we increase the energy ($E < 0$ thus reducing $|E|$), while reducing the energy (increasing $|E|$) is allowed only to a minimal value of $-2gmM/a$ which is the minimal possible energy of the system.

7 Sliding Pendulum - Bonus

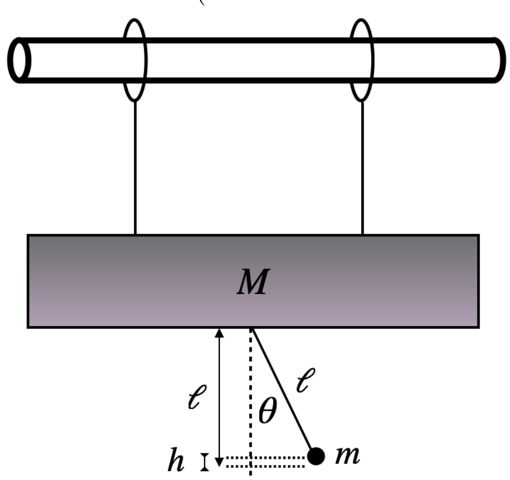
A body with mass M is hanged by from two massless loops on a frictionless cylinder. A massless rod with length ℓ connects between the center of the body and a ball with mass m, as shown in the figure. The ball is released from an unknown angle and the system begins to move.



1. Find the ratio between the horizontal velocity of mass M and that of mass m. (*Hint*: treat both masses as one system)
2. Find an expression for the kinetic and potential energy of the system, as a function of the angle θ and its time derivative $\dot{\theta}$ (and the other given quantities of the system: m, M and ℓ). (*Hint*: use the ratio you found)
3. Use energy conservation to show that for small angles the system follows the equation: $\ddot{\theta} + \omega^2\theta = 0$, and find ω.

Solution:

1. There are no external forces on the horizontal direction, therefore momentum is conserved in this direction
$$p_i = p_f \quad \rightarrow \quad 0 = MV_x + mv_x \quad \rightarrow \quad \frac{V_x}{v_x} = -\frac{m}{M}.$$
2. The potential energy is only due to the mass m (since M remains in the same height).



Expressing the potential energy in terms of the angle θ reads
$$E_p = mg\ell(1 - \cos\theta).$$

In order to find the total kinetic energy we start from the motion of m relative to M, which is a circular movement with constant radius $r=\ell$. Therefore, the relative velocity is $\ell\dot{\theta}\left(\cos\theta\,\hat{\boldsymbol{x}}+\sin\theta\,\hat{\boldsymbol{y}}\right)$, and the velocity relative to stationary observer is

$$\boldsymbol{v}=V_x\hat{\boldsymbol{x}}+\ell\dot{\theta}\left(\cos\theta\,\hat{\boldsymbol{x}}+\sin\theta\,\hat{\boldsymbol{y}}\right).$$

But we already know the ratio V_x/v_x, thus

$$v_x=\ell\dot{\theta}\cos\theta+V_x=\ell\dot{\theta}\cos\theta-\frac{m}{M}v_x\quad\rightarrow\quad v_x=\frac{M}{m+M}\ell\dot{\theta}\cos\theta$$

$$V=V_x=-\frac{m}{m+M}\ell\dot{\theta}\cos\theta.$$

The kinetic energy is therefore

$$\begin{aligned}E_k&=\frac{1}{2}m\left(v_x^2+v_y^2\right)+\frac{1}{2}MV^2\\&=\frac{1}{2}m\left(\frac{M^2}{(m+M)^2}\ell^2\dot{\theta}^2\cos^2\theta+\ell^2\dot{\theta}^2\sin^2\theta\right)+\frac{1}{2}M\frac{m^2}{(m+M)^2}\ell^2\dot{\theta}^2\cos^2\theta\\&=\frac{1}{2}\frac{mM}{(m+M)}\ell^2\dot{\theta}^2\cos^2\theta+\frac{1}{2}m\ell^2\dot{\theta}^2\sin^2\theta,\end{aligned}$$

and the total energy is

$$E=mg\ell(1-\cos\theta)+\frac{1}{2}\frac{mM}{(m+M)}\ell^2\dot{\theta}^2\cos^2\theta+\frac{1}{2}m\ell^2\dot{\theta}^2\sin^2\theta.$$

3. For small angles

$$\sin\theta\simeq\theta\quad\text{and}\quad\cos\theta\simeq1-\frac{1}{2}\theta^2,$$

therefore the total energy is

$$\begin{aligned}E&\simeq\frac{1}{2}mg\ell\theta^2+\frac{1}{2}\frac{mM}{(m+M)}\ell^2\dot{\theta}^2\left(1-\frac{1}{2}\theta^2\right)^2+\frac{1}{2}m\ell^2\dot{\theta}^2\theta^2\\&\simeq\frac{1}{2}mg\ell\theta^2+\frac{1}{2}\frac{mM}{(m+M)}\ell^2\dot{\theta}^2,\end{aligned}$$

which is conserved:

$$\dot{E}=mg\ell\theta\dot{\theta}+\frac{mM}{(m+M)}\ell^2\dot{\theta}\ddot{\theta}=0\quad\rightarrow\quad\ddot{\theta}+\frac{m+M}{M}\frac{g}{\ell}\dot{\theta}=0,$$

thus

$$\omega=\sqrt{\frac{m+M}{M}\frac{g}{\ell}}.$$