Midterm: Quantum Mechanics III, 2020-2021

Lecturer: Eytan Grosfeld, TA: Yarden Flitter

Physics Department, Ben-Gurion University of the Negev

A particle with mass m propagating in two dimensions, is prepared at point $(x, y) = (0, 0)$ at time $t = 0$. In the following we denote states at position (x, y) and time t by $|x, y; t\rangle$.

1. The particle experiences a time-dependent force $F(t)\hat{x}$. Write a path integral expression for the amplitude $\langle x_f, 0; T|0, 0; 0\rangle$. [50 points]

   Hint: follow the usual prescription of breaking the path into a classical part and fluctuations.

2. For the case that the force is zero, $F(t) = 0$, use the path integral formalism and your results from (1) above to:

   (a) find the normalized matrix elements of the position operators $\hat{x}$ and $\hat{y}$ at time t_1 [30 points]

   $$\frac{\langle x_f, 0; T|\hat{x}(t_1)|0, 0; 0\rangle}{\langle x_f, 0; T|0, 0; 0\rangle}, \quad \frac{\langle x_f, 0; T|\hat{y}(t_1)|0, 0; 0\rangle}{\langle x_f, 0; T|0, 0; 0\rangle}, \quad T \geq t_1 \geq 0,$$

   Hint: First calculate for $\hat{x}(t_1)$ and based on this result conclude the result for $\hat{y}(t_1)$.

   (b) find the normalized matrix elements of the products of two position operators at times t_1 and t_2 [20 points]

   $$\frac{\langle x_f, 0; T|\hat{x}(t_2)\hat{x}(t_1)|0, 0; 0\rangle}{\langle x_f, 0; T|0, 0; 0\rangle}, \quad \frac{\langle x_f, 0; T|\hat{y}(t_2)\hat{y}(t_1)|0, 0; 0\rangle}{\langle x_f, 0; T|0, 0; 0\rangle}, \quad T \geq t_2 \geq t_1 \geq 0.$$

   Explain your results for both (a) and (b).

   Hint: Recall $\int_0^t \delta(t' - t_1)dt' = \Theta(t - t_1)$ where $\Theta(t)$ is the Heaviside (step) function.

Quantum Mechanics 3 - Midterm 2020 Solution

Solution to item 1

The propagator for a particle in 2D is a 2D path integral over the x and y coordinates:

$$\langle x_f, 0; T|0, 0; 0\rangle = \int_{x(t=0)=0}^{x(t=T)=x_f} \int_{y(t=0)=0}^{y(t=T)=0} \mathcal{D}x\,(t)\,\mathcal{D}y\,(t)\,\mathrm{e}^{iS_F/\hbar}. \tag{1}$$

The Lagrangian of a particle under a force $F\,(t)\,\hat{x}$ is

$$L_F = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + F\,(t)\,x. \tag{2}$$

The classical equations of motion are

$$m\ddot{x}_{\mathrm{cl}} = F\,(t)\,, \qquad m\ddot{y}_{\mathrm{cl}} = 0. \tag{3}$$

Because of boundary conditions, it is easy to see that $y_{\mathrm{cl}}\,(t) = 0$. Let us now make a change of coordinates in the Lagrangian of Eq. (2):

$$x\,(t) = x_{\mathrm{cl}}\,(t) + \delta x\,(t)\,, \qquad y\,(t) = y_{\mathrm{cl}}\,(t) + \delta y\,(t)\,, \tag{4}$$

where $\delta x\,(t)$, $\delta y\,(t)$ are the quantum fluctuations which vanish at the end points. Then the Lagrangian becomes

$$\begin{aligned} L_F &= \frac{1}{2}m\dot{x}_{\mathrm{cl}}^2 + \frac{1}{2}m\dot{y}_{\mathrm{cl}}^2 + F\,(t)\,x_{\mathrm{cl}} \\ &+ \frac{1}{2}m\delta\dot{x}^2 + \frac{1}{2}m\delta\dot{y}^2 \\ &+ m\dot{x}_{\mathrm{cl}}\delta\dot{x} + m\dot{y}_{\mathrm{cl}}\delta\dot{y} + F\,(t)\,\delta x. \end{aligned} \tag{5}$$

Just like in class exercise 3, the third line gives zero contribution to the path integral. This is because

$$m\dot{x}_{\mathrm{cl}}\delta\dot{x} + m\dot{y}_{\mathrm{cl}}\delta\dot{y} + F\,(t)\,\delta x = m\frac{d}{dt}\left(\dot{x}_{\mathrm{cl}}\delta x + \dot{y}_{\mathrm{cl}}\delta y\right) - \left(m\ddot{x}_{\mathrm{cl}} - F\,(t)\right)\delta x - m\ddot{y}_{\mathrm{cl}}\delta y = m\frac{d}{dt}\left(\dot{x}_{\mathrm{cl}}\delta x + \dot{y}_{\mathrm{cl}}\delta y\right), \tag{6}$$

where the last equality follows from the equations of motion (Eq. 3). Thus, the last line of Eq. (5) is a total derivative of a function that vanishes at the end points. Therefore, we may write the action as

$$S_F = \int_0^T \left[\frac{1}{2}m\dot{x}_{\mathrm{cl}}^2 + F\,(t)\,x_{\mathrm{cl}}\right]dt + \int_0^T \left[\frac{1}{2}m\left(\delta\dot{x}^2 + \delta\dot{y}^2\right)\right]dt = S_{\mathrm{cl},F} + \int_0^T \left[\frac{1}{2}m\left(\delta\dot{x}^2 + \delta\dot{y}^2\right)\right]dt \tag{7}$$

Now, the path integral over the δx, δy coordinates is

$$\langle x_f, 0; T|0, 0; 0\rangle = \mathrm{e}^{iS_{\mathrm{cl},F}/\hbar} \int_{\delta x(t=0)=0}^{\delta x(t=T)=0} \int_{\delta y(t=0)=0}^{\delta y(t=T)=0} \mathcal{D}\delta x\,(t)\,\mathcal{D}\delta y\,(t)\exp\left[\frac{i}{\hbar}\int_0^T \left[\frac{1}{2}m\left(\delta\dot{x}^2 + \delta\dot{y}^2\right)\right]dt\right]. \tag{8}$$

Thus we see that the propagator can be decomposed to the classical action component and the quantum fluctuations (QF) component:

$$\boxed{\langle x_f, 0; T | 0, 0; 0\rangle = \mathrm{QF}_{\mathrm{2D}} \times \mathrm{e}^{iS_{\mathrm{cl},F}/\hbar}}, \tag{9}$$

where the quantum fluctuations in 2D are the square of the quantum fluctuations in 1D for a free particle:

$$\boxed{\mathrm{QF}_{\mathrm{2D}} = \mathrm{QF}_{\mathrm{1D}} \times \mathrm{QF}_{\mathrm{1D}} = \sqrt{\frac{m}{2\pi i\hbar T}} \times \sqrt{\frac{m}{2\pi i\hbar T}} = \frac{m}{2\pi i\hbar T}}. \tag{10}$$

The classical action is given by

$$\begin{aligned} S_{\mathrm{cl},F} &= \int_0^T \left[\frac{1}{2}m\dot{x}_{\mathrm{cl}}^2 + F(t)\,x_{\mathrm{cl}}\right] dt = \frac{1}{2}m\dot{x}_{\mathrm{cl}}x_{\mathrm{cl}}\Big|_0^T + \int_0^T \left[-\frac{1}{2}m\ddot{x}_{\mathrm{cl}} + F(t)\right] x_{\mathrm{cl}}dt \\ &= \frac{1}{2}m\dot{x}_{\mathrm{cl}}(t=T)\,x_f + \frac{1}{2}\int_0^T F(t)\,x_{\mathrm{cl}}(t)\,dt, \end{aligned} \tag{11}$$

where the second equality follows from integration by parts and the third follows from the equation of motion (Eq. 3). We only need now to find $x_{\mathrm{cl}}(t)$. In order to do so, we integrate the equation of motion twice:

$$\dot{x}_{\mathrm{cl}}(t) = \frac{1}{m}\int_0^t F(t')\,dt' + v_0 \tag{12}$$

$$x_{\mathrm{cl}}(t) = \frac{1}{m}\int_0^t dt' \int_0^{t'} F(t'')\,dt'' + v_0 t + x_0, \tag{13}$$

where x_0 and v_0 are constants of integration. Since $x(t=0)=0$ we see that $x_0 = 0$. Since $x(t=T) = x_f$ we find that

$$v_0 = \frac{x_f}{T} - \frac{1}{mT}\int_0^T dt' \int_0^{t'} F(t'')\,dt'' \tag{14}$$

Thus

$$\dot{x}_{\mathrm{cl}}(t) = \frac{x_f}{T} + \frac{1}{m}\int_0^t F(t')\,dt' - \frac{1}{mT}\int_0^T dt' \int_0^{t'} F(t'')\,dt'' \tag{15}$$

$$x_{\mathrm{cl}}(t) = x_f\frac{t}{T} + \frac{1}{m}\left[\int_0^t dt' \int_0^{t'} F(t'')\,dt'' - \frac{t}{T}\int_0^T dt' \int_0^{t'} F(t'')\,dt''\right] \tag{16}$$

After plugging $x_{\mathrm{cl}}(t)$ and $\dot{x}_{\mathrm{cl}}(t)$ in the expression we have for the classical action (Eq. 11), we get

$$\boxed{\begin{aligned} S_{\mathrm{cl},F} = \frac{mx_f^2}{2T} &+ \frac{x_f}{2}\int_0^T F(t')\,dt' - \frac{x_f}{2T}\int_0^T dt' \int_0^{t'} F(t'')\,dt'' + \frac{x_f}{2T}\int_0^T F(t)\,t dt \\ &- \frac{1}{2mT}\left(\int_0^T F(t)\,t dt\right)\left(\int_0^T dt' \int_0^{t'} F(t'')\,dt''\right) + \frac{1}{2m}\int_0^T F(t)\left(\int_0^t dt' \int_0^{t'} F(t'')\,dt''\right) dt. \end{aligned}} \tag{17}$$

Solution to item 2a

As we saw in class, we can evaluate any correlation function by using the functional derivative.

$$\begin{aligned}
\langle x_f,0;T|\hat{x}(t_1)|0,0;0\rangle &= \int_{\delta x(t=0)=0}^{\delta x(t=T)=0}\mathcal{D}\delta x(t)\int_{\delta y(t=0)=0}^{\delta y(t=T)=0}\mathcal{D}\delta y(t)\,x(t_1)\,\mathrm{e}^{iS_{F=0}/\hbar} \\
&= \int_{\delta x(t=0)=0}^{\delta x(t=T)=0}\mathcal{D}\delta x(t)\int_{\delta y(t=0)=0}^{\delta y(t=T)=0}\mathcal{D}\delta y(t)\,\frac{\hbar}{i}\frac{\delta}{\delta F(t_1)}\mathrm{e}^{iS_F/\hbar}\Big|_{F=0} \\
&= \int_{\delta x(t=0)=0}^{\delta x(t=T)=0}\mathcal{D}\delta x(t)\int_{\delta y(t=0)=0}^{\delta y(t=T)=0}\mathcal{D}\delta y(t)\,\mathrm{e}^{\frac{i}{\hbar}\int_0^T\left[\frac{1}{2}m\left(\delta\dot{x}^2+\delta\dot{y}^2\right)\right]dt}\,\frac{\hbar}{i}\frac{\delta}{\delta F(t_1)}\mathrm{e}^{iS_{\mathrm{cl},F}/\hbar}\Big|_{F=0} \\
&= \mathrm{QF}_{2\mathrm{D}}\times\mathrm{e}^{iS_{\mathrm{cl},F=0}/\hbar}\,\frac{\delta S_{\mathrm{cl},F}}{\delta F(t_1)}\Big|_{F=0}
\end{aligned} \tag{18}$$

Thus,

$$\frac{\langle x_f,0;T|\hat{x}(t_1)|0,0;0\rangle}{\langle x_f,0;T|0,0;0\rangle} = \frac{\delta S_{\mathrm{cl},F}}{\delta F(t_1)}\Big|_{F=0} \tag{19}$$

We now need to calculate the functional derivative of $S_{\mathrm{cl},F}$. In order to do so, remember that $\delta F(t)/\delta F(t_1)=\delta(t-t_1)$ and that the functional derivative of a product follows the same rule of ordinary derivative of a product, i.e. $(f\cdot g)'=f'g+fg'$. We will also use the identity from the supplied hint, $\int_0^t\delta(t'-t_1)\,dt'=\Theta(t-t_1)$.

$$\begin{aligned}
\frac{\delta S_{\mathrm{cl},F}}{\delta F(t_1)} &= \frac{x_f}{2}-\frac{x_f}{2T}\int_0^T dt'\Theta(t'-t_1)+\frac{x_ft_1}{2T}-\frac{t_1}{2mT}\int_0^T dt'\int_0^{t'}F(t'')\,dt'' \\
&\quad -\frac{1}{2mT}\int_0^T F(t)\,tdt\int_0^T dt'\Theta(t'-t_1)+\frac{1}{2m}\int_0^{t_1}dt'\int_0^{t'}F(t'')\,dt'' \\
&\quad +\frac{1}{2m}\int_0^T F(t)\left(\int_0^t dt'\Theta(t'-t_1)\right)dt \\
&= \frac{x_f}{2}-\frac{x_f(T-t_1)}{2T}+\frac{x_ft_1}{2T}-\frac{t_1}{2mT}\int_0^T dt'\int_0^{t'}F(t'')\,dt''-\frac{T-t_1}{2mT}\int_0^T F(t)\,tdt \\
&\quad +\frac{1}{2m}\int_0^{t_1}dt'\int_0^{t'}F(t'')\,dt''+\frac{1}{2m}\int_0^T F(t)\,\Theta(t-t_1)(t-t_1)\,dt,
\end{aligned} \tag{20}$$

and after plugging $F=0$ we get

$$\boxed{\frac{\langle x_f,0;T|\hat{x}(t_1)|0,0;0\rangle}{\langle x_f,0;T|0,0;0\rangle}=x_f\frac{t_1}{T}}. \tag{21}$$

We can repeat the same calculations for $y(t_1)$ but there is no need to do so. The symmetry between x and y in this problem tells us that the answer must have the same structure.

$$\boxed{\frac{\langle x_f,0;T|\hat{y}(t_1)|0,0;0\rangle}{\langle x_f,0;T|0,0;0\rangle}=y_f\frac{t_1}{T}=0}. \tag{22}$$

The solutions we obtained are nothing more than the classical trajectory of the particle.

Solution to item 2b

Next, we need to calculate the 2-point correlation function. Very similarly, it is given by:

$$\begin{aligned}\langle x_f,0;T|\hat{x}(t_2)\,\hat{x}(t_1)\,|0,0;0\rangle &= \mathrm{QF}_{2\mathrm{D}}\times\left(\frac{\hbar}{i}\right)^2\frac{\delta^2}{\delta F(t_1)\,\delta F(t_2)}\mathrm{e}^{iS_{\mathrm{cl},F}/\hbar}\Big|_{F=0}\\ &= \mathrm{QF}_{2\mathrm{D}}\times\mathrm{e}^{iS_{\mathrm{cl},F=0}/\hbar}\left[\frac{\delta S_{\mathrm{cl},F}}{\delta F(t_2)}\frac{\delta S_{\mathrm{cl},F}}{\delta F(t_1)}+\frac{\hbar}{i}\frac{\delta^2 S_{\mathrm{cl},F}}{\delta F(t_1)\,\delta F(t_2)}\right]_{F=0}\end{aligned}\tag{23}$$

Thus,

$$\frac{\langle x_f,0;T|\hat{x}(t_2)\,\hat{x}(t_1)\,|0,0;0\rangle}{\langle x_f,0;T|0,0;0\rangle}=\left[\frac{\delta S_{\mathrm{cl},F}}{\delta F(t_2)}\frac{\delta S_{\mathrm{cl},F}}{\delta F(t_1)}+\frac{\hbar}{i}\frac{\delta^2 S_{\mathrm{cl},F}}{\delta F(t_1)\,\delta F(t_2)}\right]_{F=0}\tag{24}$$

The second functional derivative of the classical action is

$$\begin{aligned}\frac{\delta^2 S_{\mathrm{cl},F}}{\delta F(t_1)\,\delta F(t_2)} &= \frac{\delta}{\delta F(t_2)}\frac{\delta S_{\mathrm{cl},F}}{\delta F(t_1)}=-\frac{t_1}{2mT}\int_0^T dt'\Theta(t'-t_2)-\frac{(T-t_1)\,t_2}{2mT}+\frac{1}{2m}\int_0^{t_1}dt'\Theta(t'-t_2)+\frac{t_2-t_1}{2m}\\ &= -\frac{t_1(T-t_2)}{2mT}-\frac{(T-t_1)\,t_2}{2mT}+\frac{t_2-t_1}{2m}=\frac{t_1t_2}{mT}-\frac{t_1+t_2}{2m}+\frac{t_2-t_1}{2m}=\frac{t_1t_2}{mT}-\frac{t_1}{m}\\ &= \frac{t_1(t_2-T)}{mT}\end{aligned}\tag{25}$$

Therefore we have

$$\boxed{\frac{\langle x_f,0;T|\hat{x}(t_2)\,\hat{x}(t_1)\,|0,0;0\rangle}{\langle x_f,0;T|0,0;0\rangle}=x_f^2\frac{t_1t_2}{T^2}+i\hbar\frac{t_1(T-t_2)}{mT}}.\tag{26}$$

and similarly,

$$\boxed{\frac{\langle x_f,0;T|\hat{y}(t_2)\,\hat{y}(t_1)\,|0,0;0\rangle}{\langle x_f,0;T|0,0;0\rangle}=y_f^2\frac{t_1t_2}{T^2}+i\hbar\frac{t_1(T-t_2)}{mT}=i\hbar\frac{t_1(T-t_2)}{mT}}.\tag{27}$$

If we define $\Delta\hat{x}(t)=\hat{x}(t)-\langle\hat{x}(t)\rangle$, $\Delta\hat{y}(t)=\hat{y}(t)-\langle\hat{y}(t)\rangle$, then we see that we get

$$\frac{\langle x_f,0;T|\Delta\hat{x}(t_2)\,\Delta\hat{x}(t_1)\,|0,0;0\rangle}{\langle x_f,0;T|0,0;0\rangle}=\frac{\langle x_f,0;T|\Delta\hat{y}(t_2)\,\Delta\hat{y}(t_1)\,|0,0;0\rangle}{\langle x_f,0;T|0,0;0\rangle}=i\hbar\frac{t_1(T-t_2)}{mT}.\tag{28}$$

As you might expect, the covariance does not depend on the end points, and it decreases strongly as t_1 and t_2 are farther away from each other. Specifically, at $t_1=t_2=t$ we get

$$\frac{\langle x_f,0;T|\left(\Delta\hat{x}(t)\right)^2|0,0;0\rangle}{\langle x_f,0;T|0,0;0\rangle}=\frac{\langle x_f,0;T|\left(\Delta\hat{y}(t)\right)^2|0,0;0\rangle}{\langle x_f,0;T|0,0;0\rangle}=i\hbar\frac{t(T-t)}{mT},\tag{29}$$

which means that the uncertainty in the particle's position (in both axes!) is greatest at $t=T/2$ and vanishes at the end points.