Gravity 1 - Solution 5

The Geodesic Equation

December 14, 2022

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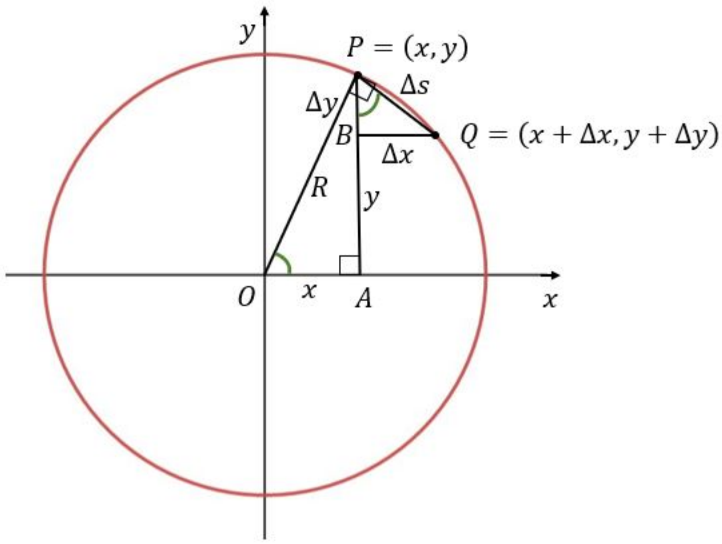

Figure 1: Circle

1 Archimedes - Lambert Projection of the Sphere

1.1 Metric on a Circle

See Figure 1. In the limit when the two points on the circle P, Q coincide, the line PQ becomes tangent to the circle, and then the angle $\angle OPQ$ approaches a right angle. Then the two green angles in the figure are equal, and the triangles ΔOPA and ΔPQB become similar. Then,

$$\frac{PQ}{PB} = \frac{OP}{OA} \tag{1}$$

that is,

$$\frac{ds}{dy} = \frac{R}{x} \tag{2}$$

$$ds = \frac{R}{x} dy \tag{3}$$

With Pythagoras theorem

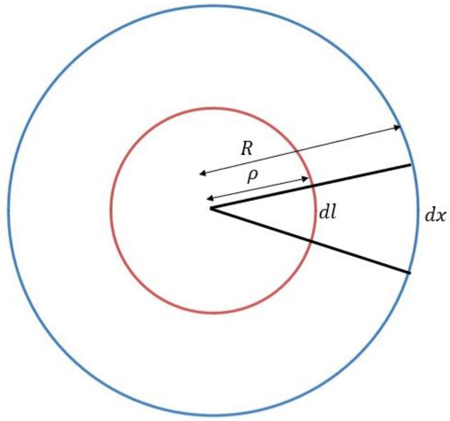

Figure 2: Top down view on a latitude of the sphere and the cylinder

$$x^2 + y^2 = R^2 \tag{4}$$

we have that

$$ds^2 = \frac{R^2}{x^2} dy^2 = \frac{R^2}{R^2 - y^2} dy^2 \tag{5}$$

1.2 The Metric

We write the line element on the sphere as a sum of line element along a longitude dk^2 and a line element along a latitude dl^2 (like we do in spherical coordinates)

$$ds^2 = dk^2 + dl^2 \tag{6}$$

The projection of the longitude element is like the metric on the circle with y coordinate. From the previous result

$$dk^2 = \frac{R^2}{R^2 - y^2} dy^2 \tag{7}$$

The projection of the latitude element dx is in proportion

$$\frac{dl}{dx} = \frac{\rho}{R} \tag{8}$$

where ρ is the distance from the z axis. See Figure 2.

$$\rho^2 = R^2 - y^2 \tag{9}$$

Therefore

$$dl^2 = \frac{R^2 - y^2}{R^2} dx^2 \tag{10}$$

In total the metric in the x, y coordinates on the cylinder is

$$ds^2 = \frac{R^2 - y^2}{R^2} dx^2 + \frac{R^2}{R^2 - y^2} dy^2 \tag{11}$$

1.3 Areas

The area element of a diagonal metric in two dimensions is

$$dA = \sqrt{g_{xx} g_{yy}} dxdy = dxdy \tag{12}$$

Therefore the area element on the map $dxdy$ (on the cylinder) is the same as the area it represents on the sphere. This map does not preserves each length separately, but it stretches and shrink the two direction exactly in with inverse factors. Therefore it is area preserving!

The area of the whole sphere then equals the area of the cylinder, i.e., the rectangle we started with

Area of a sphere

$$S = (2\pi R)(2R) = 4\pi R^2 \tag{13}$$

The area of a circle with center at the north pole and spherical radius r then equals the area on the cylinder with height

$$h = R - R\cos\theta = R\left(1 - \cos\left(\frac{r}{R}\right)\right) \tag{14}$$

Area of a spherical circle

$$A(r) = 2\pi h = 2\pi R\left(1 - \cos\left(\frac{r}{R}\right)\right) \tag{15}$$

1.4 Christoffel Symbols and the Geodesic Equation

We use the formulas for a diagonal metric.

$$g_{xx} = \frac{R^2 - y^2}{R^2} \quad , \quad g_{yy} = \frac{R^2}{R^2 - y^2} \tag{16}$$

$$\Gamma^x_{yy} = -\frac{1}{2} g_{xx}^{-1} \partial_x g_{yy} = 0 \tag{17}$$

$$\Gamma^y_{xx} = -\frac{1}{2} g_{yy}^{-1} \partial_y g_{xx} = -\frac{1}{2} \frac{R^2 - y^2}{R^2} \left(-\frac{2y}{R^2} \right) = \frac{\left(R^2 - y^2\right) y}{R^4} \tag{18}$$

$$\Gamma^x_{xy} = \frac{1}{2} g_{xx}^{-1} \partial_y g_{xx} = \frac{1}{2} \frac{R^2}{R^2 - y^2} \left(-\frac{2y}{R^2} \right) = \frac{-y}{R^2 - y^2} \tag{19}$$

$$\Gamma^y_{yx} = \frac{1}{2} g_{yy}^{-1} \partial_x g_{yy} = 0 \tag{20}$$

$$\Gamma^x_{xx} = 0 \tag{21}$$

$$\Gamma^y_{yy} = \frac{1}{2} g_{yy}^{-1} \partial_y g_{yy} = \frac{1}{2} \frac{R^2 - y^2}{R^2} \left(\frac{-R^2}{\left(R^2 - y^2\right)^2} \right) (-2y) = \frac{y}{R^2 - y^2} \tag{22}$$

In summary:

$$\boxed{\Gamma^y_{xx} = \frac{\left(R^2 - y^2\right) y}{R^4} \quad , \quad \Gamma^x_{xy} = \Gamma^x_{yx} = \frac{-y}{R^2 - y^2} \quad , \quad \Gamma^y_{yy} = \frac{y}{R^2 - y^2}} \tag{23}$$

The geodesic equations are

$$\frac{d^2 x}{d\tau^2} - \frac{2y}{R^2 - y^2} \frac{dx}{d\tau} \frac{dy}{d\tau} = 0 \tag{24}$$

$$\frac{d^2 y}{d\tau^2} + \frac{\left(R^2 - y^2\right) y}{R^4} \left(\frac{dx}{d\tau} \right)^2 + \frac{y}{R^2 - y^2} \left(\frac{dy}{d\tau} \right)^2 = 0 \tag{25}$$

One geodesic solution for both equations is $y = 0$, $x = a\tau + b$, which represents the equator of the sphere, indeed a great circle.

2 Stereographic Projection of the Sphere

2.1 Coordinates Transformation

We write the x, y, z coordinates of the three points under consideration. The north pole is at $N = (0, 0, R)$, a point on the sphere is at $P = (x, y, z)$, and its projection in the equatorial plane is at $s(P) = (x', y', 0)$. Let us find the parametric representation of the line joining these points. A direction vector is $\overrightarrow{Ns(P)}$=$(x', y', -R)$, and a position vector on the line is $\overrightarrow{N}$. Therefore the line had a parameterization

$$l : \overrightarrow{N} + t\overrightarrow{Ns(P)} = (0, 0, R) + t(x', y', -R) = (tx', ty', R - tR) \tag{26}$$

with some parameter t. The point P is on the line and on the sphere, therefore

$$x^2 + y^2 + z^2 = (tx')^2 + (ty')^2 + (R - tR)^2 = R^2 \tag{27}$$

$$t^2x'^2 + t^2y'^2 + R^2 - 2tR^2 + t^2R^2 = R^2 \tag{28}$$

$$t\left(t\left(x'^2 + y'^2 + R^2\right) - 2R^2\right) = 0 \tag{29}$$

$t = 0$ is the north pole, and the second root gives the point P.

$$t = \frac{2R^2}{R^2 + x'^2 + y'^2} \tag{30}$$

$$x(x', y') = \frac{2R^2x'}{R^2 + x'^2 + y'^2} \tag{31}$$

$$y(x', y') = \frac{2R^2y'}{R^2 + x'^2 + y'^2} \tag{32}$$

$$z(x', y') = R - \frac{2R^3}{R^2 + x'^2 + y'^2} \tag{33}$$

2.2 The Metric in Stereographic Coordinates

We write in short the relations

$$r'^2 = x'^2 + y'^2 \tag{34}$$

$$R - z = \frac{2R^3}{R^2 + r'^2} \tag{35}$$

$$x = \frac{1}{R}x'(R - z) \tag{36}$$

$$y = \frac{1}{R}y'(R - z) \tag{37}$$

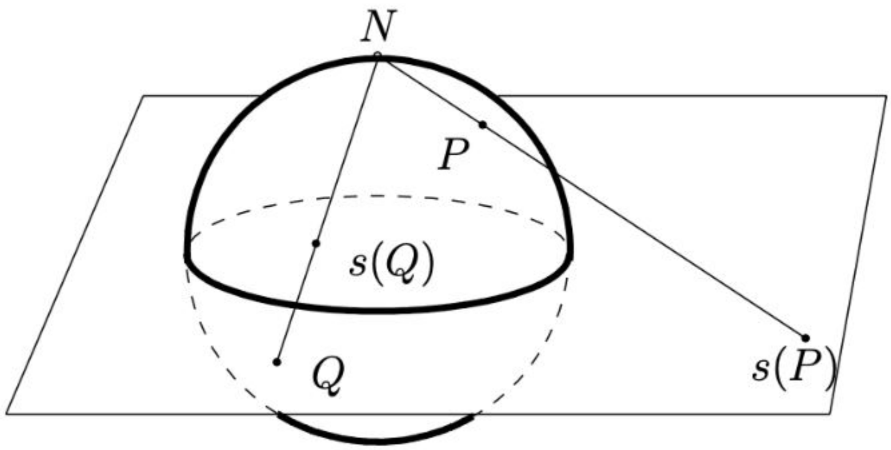

Figure 3: Stereographic Projection

The differentials

$$dz = \frac{4R^3}{\left(R^2 + r'^2\right)^2} r' dr' = \frac{2\left(R - z\right) r' dr'}{R^2 + r'^2} \tag{38}$$

$$dx = \frac{1}{R} dx' \left(R - z\right) - \frac{1}{R} x' dz \tag{39}$$

$$dy = \frac{1}{R} dy' \left(R - z\right) - \frac{1}{R} y' dz$$

Plug into the Euclidean metric

$$
\begin{aligned}
ds^2 =& dx^2 + dy^2 + dz^2 = \left(\frac{1}{R}dx'\left(R-z\right) - \frac{1}{R}x'dz\right)^2 + \left(\frac{1}{R}dy'\left(R-z\right) - \frac{1}{R}y'dz\right)^2 + dz^2 \\
=& \frac{1}{R^2}\left(dx'\right)^2\left(R-z\right)^2 + \frac{1}{R^2}x'^2dz^2 - \frac{2}{R^2}\left(R-z\right)x'dx'dz \\
+& \frac{1}{R^2}\left(dy'\right)^2\left(R-z\right)^2 + \frac{1}{R^2}y'^2dz^2 - \frac{2}{R^2}\left(R-z\right)y'dy'dz + dz^2 \\
=& \frac{1}{R^2}\left(R-z\right)^2\left(\left(dx'\right)^2 + \left(dy'\right)^2\right) - \frac{2}{R^2}\left(R-z\right)\left(x'dx' + y'dy'\right)dz + \left(1 + \frac{x'^2+y'^2}{R^2}\right)dz^2 \\
=& \frac{1}{R^2}\left(R-z\right)^2 ds_E^2 - \frac{2}{R^2}\left(R-z\right)r'dr'dz + \left(\frac{R^2+r'^2}{R^2}\right)dz^2 \\
=& \frac{1}{R^2}\left(R-z\right)^2 ds_E^2 - \frac{1}{R^2}\left(R^2+r'^2\right)dz^2 + \left(\frac{R^2+r'^2}{R^2}\right)dz^2 \\
=& \frac{1}{R^2}\left(\frac{2R^3}{R^2+r'^2}\right)^2 ds_E^2
\end{aligned}
\tag{40}
$$

$$
ds^2 = \frac{4R^4}{\left(R^2+x'^2+y'^2\right)^2}\left(dx'^2 + dy'^2\right) \tag{41}
$$

Sphere metric in stereographic coordinates

The metric is a function times the Euclidean metric. It has the same factor in front of dx'^2 and dy'^2 so the ratio of the infinitesimals is remained unchanged, i.e. it preserves angles. The metric is said to be confomally flat.

2.3 Christoffel Symbols and the Geodesic Equation

The metric in polar coordinates reads (we drop the $'$ signs now)

$$
ds^2 = \frac{4R^4}{\left(R^2+r^2\right)^2}\left(dr^2 + r^2d\theta^2\right) \tag{42}
$$

The Lagrangian is

$$
L = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \frac{1}{2}\left[g_{rr}\dot{r}^2 + g_{\theta\theta}\dot{\theta}^2\right] = \frac{1}{2}\left[g_{rr}\dot{r}^2 + g_{\theta\theta}\dot{\theta}^2\right] \tag{43}
$$

$$
L = \frac{2R^4}{\left(R^2+r^2\right)^2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \tag{44}
$$

Equation for θ We will use

$$\frac{d}{dr}\left(\frac{1}{\left(R^2+r^2\right)^2}\right)=\frac{-4r}{\left(R^2+r^2\right)^3} \tag{45}$$

$$\frac{\partial L}{\partial \theta}=0 \tag{46}$$

$$\frac{\partial L}{\partial \dot{\theta}}=\frac{4R^4}{\left(R^2+r^2\right)^2}r^2\dot{\theta} \tag{47}$$

$$\begin{aligned}\frac{d}{d\tau}\frac{\partial L}{\partial \dot{\theta}}=&\frac{\partial}{\partial r}\left(\frac{4R^4}{\left(R^2+r^2\right)^2}r^2\right)\dot{r}\dot{\theta}+\frac{4R^4}{\left(R^2+r^2\right)^2}r^2\ddot{\theta}\\ =&4R^4\left[\left(\frac{-4r}{\left(R^2+r^2\right)^3}r^2+\frac{1}{\left(R^2+r^2\right)^2}2r\right)\dot{r}\dot{\theta}+\frac{1}{\left(R^2+r^2\right)^2}r^2\ddot{\theta}\right]\\ =&4R^4\left[\left(\frac{2r\left(R^2-r^2\right)}{\left(R^2+r^2\right)^3}\right)\dot{r}\dot{\theta}+\frac{1}{\left(R^2+r^2\right)^2}r^2\ddot{\theta}\right]\end{aligned} \tag{48}$$

Then the equation of motion is

$$\left(\frac{2r\left(R^2-r^2\right)}{\left(R^2+r^2\right)^3}\right)\dot{r}\dot{\theta}+\frac{1}{\left(R^2+r^2\right)^2}r^2\ddot{\theta}=0 \tag{49}$$

$$\ddot{\theta}+\frac{2\left(R^2-r^2\right)}{r\left(R^2+r^2\right)}\dot{r}\dot{\theta}=0$$

Geodesic equation for θ

We read off the non-vanishing Christoffel symbols with upper θ index as the coefficients of the velocities terms

$$\Gamma^{\theta}_{\theta r}=\Gamma^{\theta}_{r\theta}=\frac{R^2-r^2}{r\left(R^2+r^2\right)}$$

Equation for r

$$\frac{\partial L}{\partial r} = 2R^4 \frac{\partial}{\partial r}\left[\frac{1}{\left(R^2+r^2\right)^2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right)\right]$$
$$= 2R^4\left[\frac{-4r}{\left(R^2+r^2\right)^3}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{1}{\left(R^2+r^2\right)^2}2r\dot{\theta}^2\right]$$
$$= 4R^4\left[\frac{-2r}{\left(R^2+r^2\right)^3}\dot{r}^2 + \frac{r\left(R^2-r^2\right)}{\left(R^2+r^2\right)^3}\dot{\theta}^2\right] \tag{50}$$

$$\frac{\partial L}{\partial \dot{r}} = \frac{4R^4}{\left(R^2+r^2\right)^2}\dot{r} \tag{51}$$

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{r}} = 4R^4\left[\frac{\partial}{\partial r}\left(\frac{1}{\left(R^2+r^2\right)^2}\right)\dot{r}^2 + \frac{1}{\left(R^2+r^2\right)^2}\ddot{r}\right]$$
$$= 4R^4\left[\frac{-4r}{\left(R^2+r^2\right)^3}\dot{r}^2 + \frac{1}{\left(R^2+r^2\right)^2}\ddot{r}\right] \tag{52}$$

The equation of motion is

$$0 = \frac{d}{d\tau}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 4R^4\left[\frac{-4r}{\left(R^2+r^2\right)^3}\dot{r}^2 + \frac{1}{\left(R^2+r^2\right)^2}\ddot{r} + \frac{2r}{\left(R^2+r^2\right)^3}\dot{r}^2 - \frac{r\left(R^2-r^2\right)}{\left(R^2+r^2\right)^3}\dot{\theta}^2\right]$$

$$\ddot{r} - \frac{2r}{R^2+r^2}\dot{r}^2 - \frac{r\left(R^2-r^2\right)}{R^2+r^2}\dot{\theta}^2 = 0 \tag{53}$$

Geodesic equation for r

We read off the non-vanishing Christoffel symbols with upper r index as the coefficients of the velocities terms

$$\Gamma^r_{rr} = \frac{-2r}{R^2+r^2} \quad , \quad \Gamma^r_{\theta\theta} = \frac{-r\left(R^2-r^2\right)}{R^2+r^2} \tag{54}$$

One geodesic solution for both equations is $r = R$, $\theta = a\tau + b$, which represents the equator of the sphere, indeed a great circle.

3 Three Dimensional Sphere

The equation of the 3-dimensional sphere is

$$x^2 + y^2 + z^2 + w^2 = R^2 \tag{55}$$

3.1 Coordinates Transformation

Define the radial coordinate in $\mathbb{R}^3$

$$r^2 = x^2 + y^2 + z^2 \tag{56}$$

Write the usual spherical coordinates (r, θ, ϕ)

$$x = r \sin\theta \cos\phi \tag{57}$$

$$y = r \sin\theta \sin\phi \tag{58}$$

$$z = r \cos\theta \tag{59}$$

The equation in the rw plane is

$$r^2 + w^2 = R^2 \tag{60}$$

With angle χ from the w-axis

$$w = R \cos\chi \tag{61}$$

$$r = R \sin\chi \tag{62}$$

Therefore the total coordinates transformations are

$$x = R \sin\chi \sin\theta \cos\phi \tag{63}$$

$$y = R \sin\chi \sin\theta \sin\phi \tag{64}$$

$$z = R \sin\chi \cos\theta \tag{65}$$

$$w = R \cos\chi \tag{66}$$

3.2 The Metric in Spherical Coordinates

We plug the coordinate transformation into the metric of 4-dimensional Euclidean space

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 + dw^2 \\ &= dr^2 + r^2 \left(d\theta^2 + \sin^2 d\phi^2\right) + dw^2 \\ &= R^2 d\chi^2 + (R\sin\chi)^2 \left(d\theta^2 + \sin^2 d\phi^2\right) \end{aligned} \tag{67}$$

$$ds^2 = R^2 \left[d\chi^2 + \sin^2\chi \left(d\theta^2 + \sin^2 d\phi^2\right)\right] \tag{68}$$

Metric of S^3 in spherical coordinates

where we used the metrics for Euclidean space in sphrical coordinates (r, θ, ϕ)

$$dx^2 + dy^2 + dz^2 = dr^2 + r^2 \left(d\theta^2 + \sin^2 d\phi^2\right) \tag{69}$$

and the metric on a circle of radius R, with angle χ

$$dr^2 + dw^2 = R^2 d\chi^2 \tag{70}$$