Gravity 1 - Tutorial 9

Covariant Derivative

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Contents

1 Transformation of Tensor Components Between Coordinate Bases

Given a coordinate transformation $x^\mu \to x^{\mu'}$ and a vector V, find the change-of-basis matrix between coordinate bases.

The vector components in the new coordinate basis are defined as

$$V^{\mu'} = \frac{dx^{\mu'}}{d\lambda} = \frac{dx^\mu}{d\lambda}\frac{\partial x^{\mu'}}{\partial x^\mu} = V^\mu \frac{\partial x^{\mu'}}{\partial x^\mu} \tag{1}$$

Therefore the vector components transform under change of coordinate basis as

$$V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} V^\mu \tag{2}$$

The (inverse) change-of-basis matrix is the *Jacobian matrix* of the coordinate transformation $J^{\mu'}_{\ \ \mu} = \frac{\partial x^{\mu'}}{\partial x^\mu}$, and the change-of-basis matrix is the inverse Jacobian matrix $J^{\mu}_{\ \ \mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}$.

The components of a covector ω transform under change of coordinate basis as

$$\omega_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \omega_\mu \tag{3}$$

The components of a $\binom{2}{1}$-tensor T transform under change of coordinate basis as

$$T^{\mu'\nu'}{}_{\rho'} = \frac{\partial x^{\mu'}}{\partial x^\mu}\frac{\partial x^{\nu'}}{\partial x^\nu}\frac{\partial x^\rho}{\partial x^{\rho'}} T^{\mu\nu}{}_\rho \tag{4}$$

<u>Verify</u> that the transformation matrices of vector (2) and covector (3) are the inverse of each other.

We multiply the matrices

$$\frac{\partial x^{\mu'}}{\partial x^\rho}\frac{\partial x^\rho}{\partial x^{\nu'}} = \frac{\partial x^{\mu'}}{\partial x^{\nu'}} = \delta^{\mu'}_{\nu'} \tag{5}$$

Therefore $\frac{\partial x^{\mu'}}{\partial x^\mu}$ and $\frac{\partial x^\mu}{\partial x^{\mu'}}$ are inverse matrices, and the vector and covector components transform the opposite way.

1.1 Transformation of the Metric Tensor Components

Find the transformation rule for the components of the metric tensor from the invariance of the line element.

$$ds^2 = g_{\mu'\nu'}dx^{\mu'}dx^{\nu'} = g_{\mu\nu}dx^{\mu}dx^{\nu}$$
$$= g_{\mu\nu}\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}}dx^{\mu'}dx^{\nu'} \tag{6}$$

Therefore

$$g_{\mu'\nu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}}g_{\mu\nu} \tag{7}$$

Transformation of the metric components

This is of course with agreement with the transformation of a $\binom{0}{2}$ tensor components.

1.2 Lorentz Transformation

Show that for Lorentz boost coordinates transformation, the transformation rule (2) leads to the familiar transformation of vector components.

The coordinates transformation of Lorentz boost along the x direction is

$$t' = \gamma t + \gamma v x \tag{8}$$
$$x' = \gamma x + \gamma v t \tag{9}$$
$$y' = y \tag{10}$$
$$z' = z \tag{11}$$

The transformation matrix is

$$J^{\mu'}_{\ \mu} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} = \begin{pmatrix} \frac{\partial t'}{\partial t} & \frac{\partial t'}{\partial x} & \frac{\partial t'}{\partial y} & \frac{\partial t'}{\partial z} \\ \frac{\partial x'}{\partial t} & \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} & \frac{\partial x'}{\partial z} \\ \frac{\partial y'}{\partial t} & \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} & \frac{\partial y'}{\partial z} \\ \frac{\partial z'}{\partial t} & \frac{\partial z'}{\partial x} & \frac{\partial z'}{\partial y} & \frac{\partial z'}{\partial z} \end{pmatrix} = \begin{pmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \Lambda^{\mu'}_{\ \mu} \tag{12}$$

Therefore the vector components transform as

$$V^{\mu'} = \Lambda^{\mu'}_{\ \mu}V^{\mu} \tag{13}$$

Remark: Any **linear** coordinate transformation can be written in short matrix notation as $x' = Ax$, where A is a constant matrix. The Jacobian matrix is just $J = \frac{\partial x'}{\partial x} = A$, i.e., the constant coefficients of the coordinates transformation. Only in this special case the coordinates themselves "transform as a vector".

1.3 Transforming from Cartesian to Spherical Coordinates

Find the components in spherical basis of the following covector field , given in Cartesian coordinates as

$$A_\mu = \left(xy, 2y - z^2, xz\right) \tag{14}$$

The coordinates transformation is

$$x = r\sin\theta\cos\phi \tag{15}$$
$$y = r\sin\theta\sin\phi \tag{16}$$
$$z = r\cos\theta \tag{17}$$

The old coordinates are $x^\mu = (x, y, z)$ and the new coordinates are $x^{\mu'} = (r, \theta, \phi)$.

We need to use formula (3). The transformation matrix is

$$J^\mu{}_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} \tag{18}$$

The components of (14) in spherical coordinates are

$$A_x = xy = r^2\sin^2\theta\sin\phi\cos\phi \tag{19}$$
$$A_y = 2y - z^2 = 2r\sin\theta\sin\phi - r^2\cos^2\phi \tag{20}$$
$$A_z = xz = r^2\sin\theta\cos\theta\cos\phi \tag{21}$$

By (3), the components of the covector field in spherical basis are

$$\begin{aligned} A_r &= \frac{\partial x}{\partial r}A_x + \frac{\partial y}{\partial r}A_y + \frac{\partial z}{\partial r}A_z \\ &= (\sin\theta\cos\phi)\left(r^2\sin^2\theta\sin\phi\cos\phi\right) + (\sin\theta\sin\phi)\left(2r\sin\theta\sin\phi - r^2\cos^2\phi\right) \\ &+ (\cos\theta)\left(r^2\sin\theta\cos\theta\cos\phi\right) \end{aligned} \tag{22}$$

$$
\begin{aligned}
A_\theta &= \frac{\partial x}{\partial \theta} A_x + \frac{\partial y}{\partial \theta} A_y + \frac{\partial z}{\partial \theta} A_z \\
&= (r\cos\theta\cos\phi)\left(r^2\sin^2\theta\sin\phi\cos\phi\right) + (r\cos\theta\sin\phi)\left(2r\sin\theta\sin\phi - r^2\cos^2\phi\right) \\
&+ (-r\sin\theta)\left(r^2\sin\theta\cos\theta\cos\phi\right)
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
A_\phi &= \frac{\partial x}{\partial \phi} A_x + \frac{\partial y}{\partial \phi} A_y + \frac{\partial z}{\partial \phi} A_z \\
&= (-r\sin\theta\sin\phi)\left(r^2\sin^2\theta\sin\phi\cos\phi\right) + (r\sin\theta\cos\phi)\left(r^2\sin\theta\cos\theta\cos\phi\right)
\end{aligned} \tag{24}
$$

2 Covariant Derivative

2.1 First Look: Transformation Property

Question: Are partial derivatives of a **function**, $\frac{\partial f}{\partial x^\mu}$, the components of a tensor field?

We check how they transform under change of coordinate basis. By chain rule

$$
\frac{\partial f}{\partial x^{\mu'}} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial f}{\partial x^\mu} \tag{25}
$$

Yes! This is the covector field components $\omega_\mu = \frac{\partial f}{\partial x^\mu}$, which transform as (3).

Question: Are partial derivatives of a **vector field**, $\frac{\partial V^\nu}{\partial x^\mu}$, the components of a tensor field?

We check how they transform under change of coordinate basis. By chain rule and (2)

$$
\frac{\partial V^{\nu'}}{\partial x^{\mu'}} = \frac{\partial}{\partial x^{\mu'}} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu}\left(\frac{\partial x^{\nu'}}{\partial x^\nu} V^\nu\right) = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \frac{\partial V^\nu}{\partial x^\mu} + \frac{\partial x^\mu}{\partial x^{\mu'}}\left(\frac{\partial}{\partial x^\mu} \frac{\partial x^{\nu'}}{\partial x^\nu}\right) V^\nu \tag{26}
$$

$$
\partial_{\mu'} V^{\nu'} = J^\mu{}_{\mu'} J^{\nu'}{}_\nu \left(\partial_\mu V^\nu\right) + \left(J^\mu{}_{\mu'} \partial_\mu J^{\nu'}{}_\nu\right) V^\nu \tag{27}
$$

No! While the first term in (27) is how $\begin{pmatrix}1\\1\end{pmatrix}$-tensor components transform, there is an additional term that "spoils" the tensor transformation law. This is not a good derivative of vector fields, since the partial derivative can be zero in one frame and non-zero in another. For example, we cannot meaningfully say that a vector field is constant in some direction.

We **define** a *directional covariant derivative* of vector fields as the partial derivative plus a "correction term", $\nabla_\mu := \partial_\mu + \Gamma_\mu$, such that $\nabla_\mu V^\nu$ do transform as $\begin{pmatrix}1\\1\end{pmatrix}$-tensor components. The Γ_μ act on a vector as matrices, one for each direction μ. That is,

$$\nabla_\mu V^\nu = \frac{\partial V^\nu}{\partial x^\mu} + \Gamma^\nu_{\mu\rho} V^\rho \tag{28}$$

Covariant derivative of a vector field

This means that $\frac{\partial V^\nu}{\partial x^\mu}$ do not transform as a tensor (as we saw), and also $\Gamma^\nu_{\mu\rho} V^\rho$ does not transform as a tensor, but the new symbol $\Gamma^\nu_{\mu\rho}$ is required to transform in such a way such that the combination $\frac{\partial V^\nu}{\partial x^\mu} + \Gamma^\nu_{\mu\rho} V^\rho$ does transform as a tensor. That is, the transformation law for $\Gamma^\nu_{\mu\rho}$ would also have a second term as in (27) so to cancel each other (homework).

2.2 Second Look: Parallel Transport

We had a problem differentiating a vector field with the partial derivative. What is the geometric reason for that? The answer is that for a derivative we need to subtract two vectors at two neighboring points, but they live in two different tangent spaces. In order to subtract the two vectors we first need to transport one into the tangent space of the other, and only then can we subtract (or add) them in order to form the derivative. The covariant derivative of a vector field $\boldsymbol{V}$ in the direction of $\boldsymbol{U}$ at the point P is defined as

$$\nabla_{\boldsymbol{U}} \boldsymbol{V}(P) := \lim_{\epsilon \to 0} \frac{\boldsymbol{V}_{\parallel}(Q \dashrightarrow P) - \boldsymbol{V}(P)}{\epsilon} \tag{29}$$

where $\boldsymbol{V}_{\parallel}(Q \dashrightarrow P)$ is the *parallel transport* of the vector $\boldsymbol{V}(Q)$ to the point P. Let $P = x^\mu$, $Q = x^\mu + \epsilon U^\mu$. In local inertial frame we know that

$$\nabla_\mu V^\nu = \frac{\partial V^\nu}{\partial x^\mu} \tag{30}$$

The components of a vector change under parallel transport. The changes of the components result from the changes in the angles the vector makes with the basis vectors. The changes in components will therefore be linear in the components themselves. In general, therefore, to first order in the displacement $dx^\mu = \epsilon U^\mu$, the components $V^\nu_{\parallel}(x^\mu)$ are the sum of two terms - the components V^ν at the displaced position and the changes in those components resulting

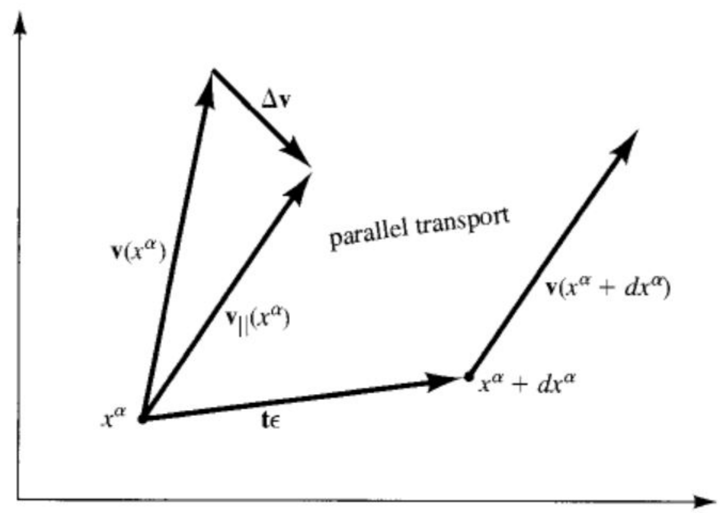

FIGURE 20.2 The derivative of a vector in flat space or in a freely falling frame in curved space. Two vectors of a vector field $\mathbf{v}(x^\alpha)$ are shown at two nearby points, x^α and $x^\alpha + dx^\alpha$, in spacetime. The two points are separated by a displacement $dx^\alpha = t^\alpha \epsilon$ along a vector t^α. To construct the difference between the vectors at x^α and $x^\alpha + dx^\alpha$, the vector $\mathbf{v}(x^\alpha + dx^\alpha)$ is first transported *parallel to itself* back to x^α to give the vector $\mathbf{v}_{\|}(x^\alpha) = [\mathbf{v}(x^\alpha + dx^\alpha)]_{\|\text{transported to } x^\alpha}$. The difference $\Delta\mathbf{v}(x^\alpha) = \mathbf{v}_{\|}(x^\alpha) - \mathbf{v}(x^\alpha)$ can then be constructed by the usual parallelogram rule. The limit $\Delta\mathbf{v}/\epsilon$ as $\epsilon \to 0$ defines the derivative of $\mathbf{v}$ in the direction of $\mathbf{t}$ at x^α.

Figure 1: Covariant derivative and parallel transport. From Hartle.

from the change in the basis vectors during parallel transport. Namely,

$$V_{\parallel}^{\nu}\left(x^{\mu}\right)=V^{\nu}\left(x^{\mu}+\epsilon U^{\mu}\right)+\Gamma_{\rho\sigma}^{\nu}\left(x^{\mu}\right)V^{\sigma}\left(x^{\mu}\right)\left(\epsilon U^{\rho}\right) \tag{31}$$

Plug in (29)

$$\begin{aligned}\nabla_{\boldsymbol{U}}V^{\nu}=&\lim_{\epsilon\to 0}\frac{V^{\nu}\left(x^{\mu}+\epsilon U^{\mu}\right)-V^{\nu}\left(x^{\mu}\right)}{\epsilon}+\lim_{\epsilon\to 0}\frac{\Gamma_{\rho\sigma}^{\nu}\left(x^{\mu}\right)V^{\sigma}\left(x^{\mu}\right)\left(\epsilon U^{\rho}\right)}{\epsilon}\\ =&U^{\mu}\frac{\partial V^{\nu}}{\partial x^{\mu}}+\Gamma_{\rho\sigma}^{\nu}\left(x^{\mu}\right)V^{\sigma}\left(x^{\mu}\right)\left(U^{\rho}\right)\end{aligned} \tag{32}$$

for $\boldsymbol{U}=e_{\mu}$ we get

$$\nabla_{\mu}V^{\nu}=\frac{\partial V^{\nu}}{\partial x^{\mu}}+\Gamma_{\mu\sigma}^{\nu}V^{\sigma} \tag{33}$$

If the transported vector coincides with vector present at the point P, i.e $\boldsymbol{V}_{\parallel}\left(Q\dashrightarrow P\right)=\boldsymbol{V}\left(P\right)$, the covariant derivative of the vector field at the point P is zero. Let $U^{\mu}=\frac{dx^{\mu}}{d\tau}$ be the tangent vector field along the curve $x^{\mu}\left(\tau\right)$. We say that the vector field $\boldsymbol{V}$ is *parallely-transported*, or *covariantly constant*, along the curve $x^{\mu}\left(\tau\right)$, if it satisfies the *parallel transport equation*

$$\nabla_{\boldsymbol{U}}\boldsymbol{V}=0 \tag{34}$$

A curve with the property that its tangent vector field $\boldsymbol{U}$ is parallely transported along itself is sometimes called an auto-parallel curve. The *auto-parallel equation* is

$$\nabla_{\boldsymbol{U}}\boldsymbol{U}=0 \tag{35}$$

Auto-parallel equation

In coordinate basis it has the following form

$$U^{\mu}\nabla_{\mu}U^{\nu}=0 \tag{36}$$

where

$$\begin{aligned}U^{\mu}\nabla_{\mu}U^{\nu}=&U^{\mu}\left(\frac{\partial U^{\nu}}{\partial x^{\mu}}+\Gamma_{\mu\sigma}^{\nu}U^{\sigma}\right)=\frac{dx^{\mu}}{d\tau}\frac{\partial U^{\nu}}{\partial x^{\mu}}+\Gamma_{\mu\sigma}^{\nu}U^{\mu}U^{\sigma}\\ =&\frac{dU^{\nu}}{d\tau}+\Gamma_{\mu\sigma}^{\nu}U^{\mu}U^{\sigma}=\frac{d^{2}x^{\nu}}{d\tau^{2}}+\Gamma_{\mu\sigma}^{\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\sigma}}{d\tau}\end{aligned} \tag{37}$$

To summarize, a covariant derivative (or *connection*) is given by information of how to parallel transport vectors, or by an assignment of the $\Gamma_{\mu\sigma}^{\nu}$ functions. This defines the notion of "straight" lines, which are the auto-parallel curves.

This structure gives the manifold a shape.

Definition: the covariant acceleration is

$$\boldsymbol{a} := \nabla_{\boldsymbol{U}}\boldsymbol{U} \tag{38}$$

Covariant acceleration vector

The auto-parallel equation for a straight line is Newton's first law

$$\boldsymbol{a} = 0 \tag{39}$$

2.3 Third Look: Axiomatic Definition

Definition: A directional covariant derivative ∇ is a map that takes a vector (field) $\boldsymbol{U}$ and a $\begin{pmatrix} p \\ q \end{pmatrix}$-tensor field T, and returns a $\begin{pmatrix} p \\ q \end{pmatrix}$-tensor (field) $\nabla_{\boldsymbol{U}}T$, satisfying the following axioms:

1. When acting on a function (a $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$-tensor field), it reduces to the directional partial derivative
$$\nabla_{\boldsymbol{U}} f = \partial_{\boldsymbol{U}} f \tag{40}$$
where $\partial_{\boldsymbol{U}} f \equiv U^{\mu}\partial_{\mu} f$.

2. Leibniz rule for the second argument
$$\nabla_{\boldsymbol{U}}(TS) = T\nabla_{\boldsymbol{U}}(S) + \nabla_{\boldsymbol{U}}(T)\,S \tag{41}$$
For example, in components, $\nabla_{\rho}(V^{\mu}W^{\nu}) = V^{\mu}\nabla_{\rho}(W^{\nu}) + \nabla_{\rho}(V^{\mu})\,W^{\nu}$.[1]

3. Additivity for the second argument
$$\nabla_{\boldsymbol{U}}(T+S) = \nabla_{\boldsymbol{U}}T + \nabla_{\boldsymbol{U}}S \tag{42}$$

4. Linearity over functions for the first argument
$$\nabla_{f\boldsymbol{U}+\boldsymbol{W}}(T) = f\nabla_{\boldsymbol{U}}(T) + \nabla_{\boldsymbol{W}}(T) \tag{43}$$
Exercise: Use these properties to express the covariant derivative of a vector field with respect to a coordinate basis

[1] There are implicit tensor products in the coordinate-free notation.

$$
\begin{aligned}
\nabla_{\boldsymbol{U}} \boldsymbol{V} = \nabla_{U^\mu e_\mu} \left(V^\rho e_\rho\right) = U^\mu \nabla_{e_\mu} \left(V^\rho e_\rho\right) \\
= & U^\mu \nabla_{e_\mu} \left(V^\rho\right) e_\rho + U^\mu V^\rho \nabla_{e_\mu} \left(e_\rho\right) \\
= & U^\mu \left(\partial_\mu V^\rho\right) e_\rho + U^\mu V^\rho \Gamma^\nu_{\mu\rho} e_\nu \\
= & U^\mu \left(\partial_\mu V^\rho + \Gamma^\rho_{\mu\nu} V^\nu\right) e_\rho
\end{aligned} \tag{44}
$$

we have

$$
\left(\nabla_\mu \boldsymbol{V}\right)^\rho = \partial_\mu V^\rho + \Gamma^\rho_{\mu\nu} V^\nu \tag{45}
$$

where we used the notation $\nabla_{e_\mu} \equiv \nabla_\mu$. The covariant derivative of the basis vector fields is another vector field which we expanded on the basis

$$
\Gamma^\nu_{\mu\rho} e_\nu := \nabla_{e_\mu} \left(e_\rho\right) \tag{46}
$$

This is the derivative of the ρ basis vector in the direction of the μ basis vector. The $\Gamma' s$ are called the *connection coefficients*, since they represent ∇ acting on a basis

Connection coefficients

$$
\boxed{\Gamma^\nu_{\mu\rho} = e^\nu \left(\nabla_{e_\mu} \left(e_\rho\right)\right)} \tag{47}
$$

They encapsulate the information/structure that we need to add in order to define the directional derivative of a vector field.

Exercise: What is the covariant derivative of a covector field? Is there additional information needed for this?

Use the Leibniz rule.

$$
\nabla_\mu \left(\omega_\nu V^\nu\right) = \nabla_\mu \left(\omega_\nu\right) V^\nu + \omega_\nu \nabla_\mu \left(V^\nu\right) \tag{48}
$$

$$
\begin{aligned}
V^\nu \nabla_\mu \omega_\nu = \nabla_\mu \left(\omega_\nu V^\nu\right) - \omega_\nu \nabla_\mu V^\nu \\
= & \partial_\mu \left(\omega_\nu V^\nu\right) - \omega_\nu \left(\partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho\right) \\
= & \partial_\mu \omega_\nu V^\nu + \omega_\nu \partial_\mu V^\nu - \omega_\nu \partial_\mu V^\nu - \Gamma^\nu_{\mu\rho} \omega_\nu V^\rho \\
= & V^\nu \left(\partial_\mu \omega_\nu - \Gamma^\rho_{\mu\nu} \omega_\rho\right)
\end{aligned} \tag{49}
$$

The ν component of the covariant derivative of a covector field ω in the direction of e_μ is

Covariant derivative of a covector field

$$
\boxed{\nabla_\mu \omega_\nu = \frac{\partial \omega_\nu}{\partial x^\mu} - \Gamma^\rho_{\mu\nu} \omega_\rho} \tag{50}
$$

The same $\Gamma^\rho_{\mu\nu}$ suffice to do the job, only with a minus sign.

By using the Leibniz rule further we get the following rule: For the covariant derivative of a tensor field we add or subtract Christoffels for each index, plus for upper index and minus for lower index. For example,

$$\nabla_\mu T^\nu{}_\rho = \frac{\partial T^\nu{}_\rho}{\partial x^\mu} + \Gamma^\nu_{\mu\sigma} T^\sigma{}_\rho - \Gamma^\sigma_{\mu\rho} T^\nu{}_\sigma \tag{51}$$

$$\nabla_\mu T_{\nu\rho} = \frac{\partial T_{\nu\rho}}{\partial x^\mu} - \Gamma^\sigma_{\mu\nu} T_{\sigma\rho} - \Gamma^\sigma_{\mu\rho} T_{\nu\sigma} \tag{52}$$

$$\nabla_\mu T^{\nu\rho} = \frac{\partial T^{\nu\rho}}{\partial x^\mu} + \Gamma^\nu_{\mu\sigma} T^{\sigma\rho} + \Gamma^\rho_{\mu\sigma} T^{\nu\sigma} \tag{53}$$

Exercise: Find the transformation law for the connection coefficients from their definition (47).

$$\begin{aligned}
\Gamma^{\rho'}_{\mu'\nu'} &= e^{\rho'}\left(\nabla_{e_{\mu'}}(e_{\nu'})\right) = J^{\rho'}{}_\rho e^\rho \left(\nabla_{J^\mu{}_{\mu'} e_\mu}\left(J^\nu{}_{\nu'} e_\nu\right)\right) \\
&= J^{\rho'}{}_\rho J^\mu{}_{\mu'} J^\nu{}_{\nu'} e^\rho \left(\nabla_{e_\mu}(e_\nu)\right) + J^{\rho'}{}_\rho J^\mu{}_{\mu'} e^\rho \left(\nabla_{e_\mu}\left(J^\nu{}_{\nu'}\right) e_\nu\right) \\
&= J^{\rho'}{}_\rho J^\mu{}_{\mu'} J^\nu{}_{\nu'} \Gamma^\rho_{\mu\nu} + J^{\rho'}{}_\rho J^\mu{}_{\mu'} \partial_\mu J^\nu{}_{\nu'} e^\rho(e_\nu) \\
&= J^{\rho'}{}_\rho J^\mu{}_{\mu'} J^\nu{}_{\nu'} \Gamma^\rho_{\mu\nu} + J^{\rho'}{}_\rho \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu J^\nu{}_{\nu'} \delta^\rho_\nu \\
&= J^{\rho'}{}_\rho J^\mu{}_{\mu'} J^\nu{}_{\nu'} \Gamma^\rho_{\mu\nu} + J^{\rho'}{}_\sigma \partial_{\mu'} J^\sigma{}_{\nu'}
\end{aligned} \tag{54}$$

$$\Gamma^{\rho'}_{\mu'\nu'} = \frac{\partial x^{\rho'}}{\partial x^\rho} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \Gamma^\rho_{\mu\nu} + \frac{\partial x^{\rho'}}{\partial x^\sigma} \frac{\partial^2 x^\sigma}{\partial x^{\mu'} \partial x^{\nu'}} \tag{55}$$

Transformation law of the Christoffel symbols

From (55) we see that $\Gamma^\rho_{\mu\nu}$ is not a tensor, but the anti-symmetric part of $\Gamma^\rho_{\mu\nu}$ (in the lower indices) is a tensor. It is called the *torsion* tensor

$$T^\rho_{\mu\nu} := \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu} \tag{56}$$

The symmetric part of $\Gamma^\rho_{\mu\nu}$ can be set to zero in suitable coordinates. The anti symmetric part cannot since it is a tensor. It is set to zero in GR by choice, $T^\rho_{\mu\nu} = 0$.

2.4 Metric Compatibility

The "straightest lines", i.e., trajectories of zero covariant acceleration at any point along the curve, are determined by the covariant derivative. They satisfy the equation

$$\frac{d^2x^\rho}{d\tau^2} + \Gamma^\rho_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{57}$$

The "shortest lines", i.e., curves that extremizes the length functional, are determined by the metric. They satisfy the equation

$$\frac{d^2x^\rho}{d\tau^2} + \left[\frac{1}{2}g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right)\right]\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{58}$$

Demanding that the "straightest lines" and the "shortest lines" coincide, means that (57) and (58) are the same equation, called the *geodesic equation*. In this case the symmetric part of the connection coefficients are given by the metric as

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) \tag{59}$$

In this case we say that the covariant derivative is compatible with the metric. If in addition there is zero torsion, the anti-symmetric part of $\Gamma^\rho_{\mu\nu}$ is zero, and (59) are the (full) Christoffel symbols.

Show that from (59) it follows that the metric is *covariantly constant*

Metric compatibility

$$\nabla_\rho g_{\mu\nu} = 0 \tag{60}$$

Notice that (60) is a third rank tensor equation, so it is zero in any basis. Equation (60) is called *metric compatibility*.

$$\begin{aligned}
\nabla_\rho g_{\mu\nu} = {} & \partial_\rho g_{\mu\nu} - \Gamma^\sigma_{\rho\mu}g_{\sigma\nu} - \Gamma^\sigma_{\rho\nu}g_{\mu\sigma} \\
= {} & \partial_\rho g_{\mu\nu} - \frac{1}{2}g^{\sigma\lambda}\left(\partial_\rho g_{\mu\lambda} + \partial_\mu g_{\rho\lambda} - \partial_\lambda g_{\rho\mu}\right)g_{\sigma\nu} - \frac{1}{2}g^{\sigma\lambda}\left(\partial_\rho g_{\nu\lambda} + \partial_\nu g_{\rho\lambda} - \partial_\lambda g_{\rho\nu}\right)g_{\mu\sigma} \\
= {} & \partial_\rho g_{\mu\nu} - \frac{1}{2}\delta^\lambda_\nu\left(\partial_\rho g_{\mu\lambda} + \partial_\mu g_{\rho\lambda} - \partial_\lambda g_{\rho\mu}\right) - \frac{1}{2}\delta^\lambda_\mu\left(\partial_\rho g_{\nu\lambda} + \partial_\nu g_{\rho\lambda} - \partial_\lambda g_{\rho\nu}\right) \\
= {} & \partial_\rho g_{\mu\nu} - \frac{1}{2}\left(\partial_\rho g_{\mu\nu} + \partial_\mu g_{\rho\nu} - \partial_\nu g_{\rho\mu}\right) - \frac{1}{2}\left(\partial_\rho g_{\nu\mu} + \partial_\nu g_{\rho\mu} - \partial_\mu g_{\rho\nu}\right) \\
= {} & 0
\end{aligned} \tag{61}$$

Geometrically, it follows that the dot product (angle) of two vectors that are parallely transported along a curve remains the same.

2.5 Covariant Differentiation on a Sphere

The non-vanishing Christoffel symbols of the sphere in (θ, ϕ) coordinates are

$$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta \tag{62}$$

$$\Gamma^{\phi}_{\phi\theta} = \Gamma^{\phi}_{\theta\phi} = \frac{\cos\theta}{\sin\theta} \tag{63}$$

1. Given the vector field V with components $V^j = (0, 1)$, calculate the four components of the tensor field $\nabla_i V^j$.

2. Show that the equator is the only latitude which is an auto-parallel. Show that all the longitudes are auto-parallels.

3. Calculate $\nabla_\theta \nabla_\phi V^\theta$ and $\nabla_\phi \nabla_\theta V^\theta$. Do the covariant derivatives commute?

2.5.1 First derivatives

The covariant derivative is

$$\nabla_i V^j = \frac{\partial V^j}{\partial x^i} + \Gamma^j_{ik} V^k = \Gamma^j_{i\theta} V^\theta + \Gamma^j_{i\phi} V^\phi = \Gamma^j_{i\phi} \tag{64}$$

since $\frac{\partial V^j}{\partial x^i} = 0$. and $V^\theta = 0, V^\phi = 1$. The four components are:

$$\nabla_\theta V^\theta = \Gamma^\theta_{\theta\phi} = 0 \tag{65}$$

$$\nabla_\theta V^\phi = \Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta} \tag{66}$$

$$\nabla_\phi V^\theta = \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \tag{67}$$

$$\nabla_\phi V^\phi = \Gamma^\phi_{\phi\phi} = 0 \tag{68}$$

2.5.2 Latitudes and Longitudes

Notice that the given vector field V is nothing but the coordinate basis vector field in the ϕ direction, $V = e_\phi$. For each θ it contains the tangent vectors to the latitude on the sphere of angle θ. The result of the previous section can be obtained from the definition of the Christoffel symbols as the connection coefficient (47). Now, an auto-parallel curve satisfies $\nabla_V V = 0$, where V is the tangent vector field to the curve. For the latitudes we found

$$(\nabla_V V)^\theta = \nabla_{e_\phi} V^\theta = \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \tag{69}$$

and

$$(\nabla_V V)^\phi = \nabla_{e_\phi} V^\phi = \Gamma^\phi_{\phi\phi} = 0 \tag{70}$$

Only for $\theta = \frac{\pi}{2}$ does $\nabla_V V = 0$, therefore the equator is the only auto-parallel latitude.

As for longitudes, the tangent vector field is e_θ. From (47) we have

$$\nabla_{e_\theta} e_\theta = \Gamma^\theta_{\theta\theta} e_\theta + \Gamma^\phi_{\theta\theta} e_\phi = 0 \tag{71}$$

for any ϕ, thus all the longitudes are auto-parallels of the sphere with respect to the given covariant derivative.

2.5.3 Second derivatives

$$\begin{aligned}\nabla_\theta \nabla_\phi V^\theta &= \frac{\partial}{\partial\theta}\left(\nabla_\phi V^\theta\right) + \Gamma^\theta_{\theta k}\nabla_\phi V^k - \Gamma^k_{\theta\phi}\nabla_k V^\theta \\ &= \frac{\partial}{\partial\theta}\left(-\sin\theta\cos\theta\right) + 0 - \Gamma^\phi_{\theta\phi}\nabla_\phi V^\theta \\ &= -\cos^2\theta + \sin^2\theta - \frac{\cos\theta}{\sin\theta}\left(-\sin\theta\cos\theta\right) = \sin^2\theta\end{aligned} \tag{72}$$

$$\begin{aligned}\nabla_\phi \nabla_\theta V^\theta &= \frac{\partial}{\partial\phi}\left(\nabla_\theta V^\theta\right) + \Gamma^\theta_{\phi k}\nabla_\theta V^k - \Gamma^k_{\phi\theta}\nabla_k V^\theta \\ &= \Gamma^\theta_{\phi\phi}\nabla_\theta V^\phi - \Gamma^\phi_{\phi\theta}\nabla_\phi V^\theta \\ &= -\sin\theta\cos\theta\frac{\cos\theta}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\left(-\sin\theta\cos\theta\right) = 0\end{aligned} \tag{73}$$

So,

$$\left(\nabla_\theta\nabla_\phi - \nabla_\phi\nabla_\theta\right) V^\theta \neq 0 \tag{74}$$

We will see next week that this is because the sphere has curvature.

2.6 Gradient, Divergence and Laplacian

1. Write down the components of gradient of a function f as a contravariant vector $\nabla^\mu f$.

2. Show that

$$\Gamma^\rho_{\rho\mu} = \frac{1}{\sqrt{g}}\partial_\mu\sqrt{g} \tag{75}$$

3. Show that the (covariant) divergence of a vector field V is given by

$$divV := \nabla_\mu V^\mu = \frac{1}{\sqrt{g}}\partial_\mu \left(\sqrt{g}V^\mu\right) \tag{76}$$

4. Show that the (covariant) Laplacian of a function f is given by

$$\nabla^2 f := \nabla_\mu \nabla^\mu f = \frac{1}{\sqrt{g}}\partial_\mu \left(\sqrt{g}g^{\mu\nu}\partial_\nu f\right) \tag{77}$$

2.6.1 Gradient

The covariant derivative acts on a function like a partial derivative.

$$\nabla_\mu f = \partial_\mu f \tag{78}$$

Raise the index with the metric

$$\nabla^\mu f = g^{\mu\nu}\partial_\nu f \tag{79}$$

Contravariant gradient

2.6.2 Contracted Christoffel

Start with the left hand side of (75).

$$\Gamma^\rho_{\rho\mu} = \frac{1}{2}g^{\rho\sigma}\left(\partial_\rho g_{\mu\sigma} + \partial_\mu g_{\rho\sigma} - \partial_\sigma g_{\rho\mu}\right) = \frac{1}{2}g^{\rho\sigma}\partial_\mu g_{\rho\sigma} \tag{80}$$

The first and last term are the same and cancel, since both are contracted as $g^{\alpha\beta}\partial_\alpha g_{\beta\mu}$ (the metric is symmetric).

To unravel the right hand side of (75) we need to differentiate the metric determinant g. Recall that for ordinary numbers the logarithm function satisfies that the logarithm of a product equals the sum of the logs, i.e.,

$$ln\left(ab\right) = ln\left(a\right) + ln\left(b\right) \tag{81}$$

A generalization exists for invertible matrices, where the product in (81) is replaced by the determinant of the matrix (the product of its eigenvalues), and the sum in (81) is replaced by the trace of the matrix (the sum of its eigenvalues). For a matrix A

$$ln\left(detA\right) = Tr\left(lnA\right) \tag{82}$$

Differentiate

$$\frac{1}{detA}\partial_\mu (detA) = Tr\left(A^{-1}\partial_\mu A\right) \tag{83}$$

For the metric $g_{\mu\nu}$, its inverse $g^{\mu\nu}$, and its determinant g we have

$$\frac{1}{g}\partial_\mu g = g^{\rho\sigma}\partial_\mu g_{\rho\sigma} \tag{84}$$

where we used one contraction as the matrix multiplication and second contraction as the trace.

Now with the square root

$$\frac{1}{\sqrt{g}}\partial_\mu\sqrt{g} = \frac{1}{\sqrt{g}}\frac{1}{2\sqrt{g}}\partial_\mu g = \frac{1}{2}\frac{1}{g}\partial_\mu g = \frac{1}{2}g^{\rho\sigma}\partial_\mu g_{\rho\sigma} \tag{85}$$

Since (80) and (85) equal the same thing, we have

$$\Gamma^\rho_{\rho\mu} = \frac{1}{\sqrt{g}}\partial_\mu\sqrt{g} \tag{86}$$

Contracted Christoffel

2.6.3 Divergence

Now we can use (86) to write the divergence.

$$\begin{aligned}\nabla_\mu V^\mu &= \partial_\mu V^\mu + \Gamma^\mu_{\mu\nu}V^\nu \\ &= \partial_\mu V^\mu + \frac{1}{\sqrt{g}}\left(\partial_\nu\sqrt{g}\right)V^\nu = \partial_\mu V^\mu + \frac{1}{\sqrt{g}}\left(\partial_\mu\sqrt{g}\right)V^\mu\end{aligned} \tag{87}$$

By the product rule for derivative we get

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}V^\mu\right) \tag{88}$$

Divergence of a vector field

2.6.4 Laplacian

The Laplacian is just the divergence of the gradient. Use (79) and (88)

$$\nabla_\mu\nabla^\mu f = \nabla_\mu\left(g^{\mu\nu}\partial_\nu f\right) = \frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu f\right) \tag{89}$$

Laplacian of a function

$$\nabla_\mu \nabla^\mu f = \frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g} g^{\mu\nu} \partial_\nu f \right) \tag{90}$$

Notice that in flat space in Cartesian coordinates, where $g_{ij} = \delta_{ij}$ and $\Gamma^k_{ij} = 0$ (so the covariant derivative reduces to the partial derivative), both sides of (79),(88) and (90) reduce to the familiar standard expressions.

3 Killing Vector Fields

3.1 Killing's Equation

Exercise: Show that if a vector field ξ satisfies *Killing's equation*

Killing's equation

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \tag{91}$$

then along geodesics with velocity vector field U, $\xi \cdot U$ is conserved.

Solution: Let us differentiate $\xi \cdot U$ along the curve

$$\begin{aligned} \frac{d}{d\tau} (\xi \cdot U) = & \frac{dx^\mu}{d\tau} \frac{\partial}{\partial x^\mu} (\xi_\nu U^\nu) = U^\mu \nabla_\mu (\xi_\nu U^\nu) \\ = & U^\mu U^\nu \nabla_\mu \xi_\nu + \xi_\nu U^\mu \nabla_\mu U^\nu \end{aligned} \tag{92}$$

where we used that the partial and covariant derivatives act the same on functions. The geodesic equation is $U^\mu \nabla_\mu U^\nu = 0$, therefore along geodesics

$$\frac{d}{d\tau} (\xi \cdot U) = U^\mu U^\nu \nabla_\mu \xi_\nu = \frac{1}{2} U^\mu U^\nu \left(\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \right) \tag{93}$$

Thus if ξ satisfies Killing's equation (91) then $\frac{d}{d\tau} (\xi \cdot U) = 0$. ξ is called a Killing vector field, and by solving (91) we can find Killing vector fields that are not obvious from the metric in specific coordinates.

Notice that in (91), ξ_μ is the dual vector field, it has an index down, as opposed to the Killing vector field components ξ^μ. We can read (91) as "the symmetric part of the second rank tensor $\nabla_\mu \xi_\nu$ vanish". Since it is a tensorial equation, it has the same form in any coordinate basis.

Exercise: Show that there is a coordinate system such that all the metric components are independent of some coordinate if and only if there is a Killing vector field tangent to this coordinate lines.

Solution: Without loss of generality, let ξ be a vector field tangent to the coordinate lines of the x^1 coordinate

$$\xi^\mu = (0, 1, 0, 0) \tag{94}$$

Compute

$$\begin{aligned}\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = & \partial_\mu \xi_\nu - \Gamma^\rho_{\mu\nu} \xi_\rho + \partial_\nu \xi_\mu - \Gamma^\rho_{\nu\mu} \xi_\rho \\ =& \partial_\mu \left(g_{\nu\rho} \xi^\rho\right) + \partial_\nu \left(g_{\mu\rho} \xi^\rho\right) - 2\Gamma^\rho_{\mu\nu} g_{\rho\sigma} \xi^\sigma \\ =& \partial_\mu g_{\nu 1} + \partial_\nu g_{\mu 1} - g_{\rho 1} g^{\rho\lambda} \left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\right) \\ =& \partial_\mu g_{\nu 1} + \partial_\nu g_{\mu 1} - \delta^\lambda_1 \left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\right) \\ =& \partial_\mu g_{\nu 1} + \partial_\nu g_{\mu 1} - \left(\partial_\mu g_{\nu 1} + \partial_\nu g_{\mu 1} - \partial_1 g_{\mu\nu}\right) \\ =& \partial_1 g_{\mu\nu}\end{aligned} \tag{95}$$

where we used

$$g_{\nu\sigma} \xi^\sigma = g_{\nu 1} \tag{96}$$

then

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = \partial_1 g_{\mu\nu} \tag{97}$$

therefore

$$\frac{\partial}{\partial x^1} g_{\mu\nu} = 0 \quad \Longleftrightarrow \quad \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \tag{98}$$

3.2 Symmetries of the Euclidean Plane

Exercise: Find a basis for the Killing vector fields of the Euclidean plane in Cartesian coordinates. What is the dimension of the Killing vector fields vector space?

The metric of the Euclidean plane in Cartesian coordinates is

$$ds^2 = dx^2 + dy^2 \tag{99}$$

and all the Christofell symbols vanish

$$\Gamma^\rho_{\mu\nu} = 0 \tag{100}$$

therefore the covariant derivatives reduce to the partial derivatives $\nabla_\mu = \partial_\mu$. Also, since the metric is $g_{ij} = \delta_{ij}$ we can raise the indices with no change of the vector components, $\xi_i = \xi^i$ for all i. We need to solve Killing's equation (91), which becomes

$$\partial_\mu \xi^\nu + \partial_\nu \xi^\mu = 0 \tag{101}$$

These are three equations

$$\partial_x \xi^x = 0 \tag{102}$$

$$\partial_y \xi^y = 0 \tag{103}$$

$$\partial_x \xi^y + \partial_y \xi^x = 0 \tag{104}$$

from (102) and (103)

$$\xi^x = \xi^x(y) \tag{105}$$

$$\xi^y = \xi^y(x) \tag{106}$$

plug in (104) and differentiate by ∂_y yields

$$\partial_y^2 \xi^x = 0 \tag{107}$$

or by ∂_x yields

$$\partial_x^2 \xi^y = 0 \tag{108}$$

therefore $\xi^x(y)$ and $\xi^y(x)$ are linear polynomials

$$\xi^x(y) = A + By \tag{109}$$

$$\xi^y(x) = C + Dx \tag{110}$$

We need to plug our solutions back into (104) since when we differentiated the equation we lost some information about the constraint. We get

$$D + B = 0 \tag{111}$$

The Killing vector fields are of the form

$$\xi^i = \begin{pmatrix} A - Dy \\ C + Dx \end{pmatrix} = A \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C \begin{pmatrix} 0 \\ 1 \end{pmatrix} + D \begin{pmatrix} -y \\ x \end{pmatrix} \tag{112}$$

The basis Killing vector fields are

Killing vector fields of the Euclidean plane

$$X^i = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad Y^i = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad R^i = \begin{pmatrix} -y \\ x \end{pmatrix} \tag{113}$$

X and Y generate translations in the x and y directions, and R generates rotations around the origin. All together these are the generators of the isometry group of the Euclidean plane They form a 3-dimensional vector space, which is the maximal dimension for symmetries of a 2-dimensional manifold.