Gravity 1 - Tutorial 10

Curvature

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1 The Geodesic Deviation Equation

1.1 Deviation Equation in Newtonian Gravity

Consider two nearby particles moving only under the influence of gravity. In an inertial frame the equation of motion for the position $x^i(t)$ of the first particle is

$$\frac{d^2 x^i}{dt^2} = -\delta^{ij} \frac{\partial \phi}{\partial x^j} \left(x^k\right) \tag{1}$$

Let S^i be a small separation vector between the two particles, $S^i \ll 1$. The position of the second particle is $x^i(t) + S^i(t)$. We write its equation of motion, and approximate to linear order in S^i

$$\frac{d^2 \left(x^i + S^i\right)}{dt^2} = -\delta^{ij} \frac{\partial \phi \left(x^k + S^k\right)}{\partial x^j} \approx -\delta^{ij} \frac{\partial \phi}{\partial x^j} \left(x^k\right) - \delta^{ij} \left(\frac{\partial}{\partial x^k} \frac{\partial \phi}{\partial x^j}\right) \left(x^k\right) S^k \tag{2}$$

Subtracting (1) from (2) yields

$$\boxed{\frac{d^2 S^i}{dt^2} = -\delta^{ij} \frac{\partial^2 \phi}{\partial x^k \partial x^j} S^k} \tag{3}$$

Newtonian deviation equation

Equation (3) is the Newtonian deviation equation. The Hessian of the scalar potential field $\frac{\partial^2 \phi}{\partial x^k \partial x^j}$ is called the *tidal tensor*. It determines the tidal forces on an extended body in a gravitational field.

1.2 The Geodesic Deviation Equation

Consider two neighboring geodesics, separated at each point by an infinitesimal separation vector S (deviation between nearby geodesics), $|S^\mu| \ll 1$. One geodesic has coordinate $x^\mu(\tau)$ and the second has coordinate $x^\mu(\tau) + S^\mu(\tau)$.

Exercise: Find the equation for the acceleration of the separation vector. This equation is called the *geodesic deviation equation*.

We denote the velocity along the first geodesic as $u^\mu = \frac{dx^\mu}{d\tau}$.

The velocity of the separation, as we move along the geodesic in the direction of the tangent u, is

$$\begin{aligned} V^\mu &= (\nabla_u S)^\mu = u^\nu \nabla_\nu S^\mu = u^\nu \partial_\nu S^\mu + u^\nu \Gamma^\mu_{\nu\rho} S^\rho \\ &= \frac{dS^\mu}{d\tau} + \Gamma^\mu_{\nu\rho} u^\nu S^\rho \end{aligned} \tag{4}$$

It is the "relative velocity of the geodesics".

The acceleration of the separation , as we move along the geodesic in the direction of the tangent u, is

$$\begin{aligned} A^\mu &= (\nabla_u V)^\mu = u^\nu \nabla_\nu V^\mu = u^\nu \partial_\nu V^\mu + u^\nu \Gamma^\mu_{\nu\rho} V^\rho \\ &= \frac{dV^\mu}{d\tau} + \Gamma^\mu_{\nu\rho} u^\nu V^\rho \end{aligned} \tag{5}$$

It is the "relative acceleration of the geodesics".

Plug (4) into (5)

$$\begin{aligned} A^\mu &= \frac{d}{d\tau}\left(\frac{dS^\mu}{d\tau} + u^\nu \Gamma^\mu_{\nu\rho} S^\rho\right) + \Gamma^\mu_{\nu\rho} u^\nu \left(\frac{dS^\rho}{d\tau} + \Gamma^\rho_{\sigma\lambda} u^\sigma S^\lambda\right) \\ &= \frac{d^2 S^\mu}{d\tau^2} + \frac{d}{d\tau}\left(u^\nu \Gamma^\mu_{\nu\rho} S^\rho\right) + \Gamma^\mu_{\nu\rho} u^\nu \frac{dS^\rho}{d\tau} + \Gamma^\mu_{\nu\rho}\Gamma^\rho_{\sigma\lambda} u^\nu u^\sigma S^\lambda \\ &= \frac{d^2 S^\mu}{d\tau^2} + \frac{du^\nu}{d\tau}\Gamma^\mu_{\nu\rho} S^\rho + u^\nu u^\lambda \partial_\lambda \Gamma^\mu_{\nu\rho} S^\rho + u^\nu \Gamma^\mu_{\nu\rho}\frac{dS^\rho}{d\tau} + \Gamma^\mu_{\nu\rho} u^\nu \frac{dS^\rho}{d\tau} + \Gamma^\mu_{\nu\rho}\Gamma^\rho_{\sigma\lambda} u^\nu u^\sigma S^\lambda \end{aligned} \tag{6}$$

Now we will write the geodesic equations for the two nearby geodesics, approximate to first order in S, and subtract them. The resulting equation (for $\frac{d^2 S^\mu}{d\tau^2}$) will be substituted into (6), as well as the equation of the first geodesic (for $\frac{du^\nu}{d\tau}$).

The equation of the first geodesic $x^\mu(\tau)$ is

$$\frac{d^2 x^\rho}{d\tau^2} + \Gamma^\rho_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{7}$$

The equation of the nearby geodesic $x^\mu(\tau) + S^\mu(\tau)$ is

$$\frac{d^2\left(x^\rho + S^\rho\right)}{d\tau^2} + \Gamma^\rho_{\mu\nu}\left(x^\sigma + S^\sigma\right)\frac{d\left(x^\mu + S^\mu\right)}{d\tau}\frac{d\left(x^\nu + S^\nu\right)}{d\tau} = 0 \tag{8}$$

To linear approximation in S^μ (8) becomes

$$\frac{d^2 x^\rho}{d\tau^2} + \frac{d^2 S^\rho}{d\tau^2} + \left(\Gamma^\rho_{\mu\nu} + S^\sigma \partial_\sigma \Gamma^\rho_{\mu\nu}\right)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + 2\Gamma^\rho_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dS^\nu}{d\tau} = 0 \tag{9}$$

where the Christoffels in (9) are evaluated on the first geodesic, to first order in separation S^σ

$$\Gamma^\rho_{\mu\nu}\left(x^\sigma + S^\sigma\right) \approx \Gamma^\rho_{\mu\nu}\left(x^\sigma\right) + S^\sigma \partial_\sigma \Gamma^\rho_{\mu\nu}\left(x^\sigma\right) \tag{10}$$

(9) can be written as

$$\frac{d^2x^\rho}{d\tau^2}+\Gamma^\rho_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}+\frac{d^2S^\rho}{d\tau^2}+2\Gamma^\rho_{\mu\nu}u^\mu\frac{dS^\nu}{d\tau}=-\partial_\sigma\Gamma^\rho_{\mu\nu}u^\mu u^\nu S^\sigma \tag{11}$$

Subtract (7) from (11) yields

$$\frac{d^2S^\rho}{d\tau^2}+2\Gamma^\rho_{\mu\nu}u^\mu\frac{dS^\nu}{d\tau}=-\partial_\sigma\Gamma^\rho_{\mu\nu}u^\mu u^\nu S^\sigma \tag{12}$$

Also, (7) can be written as

$$\frac{du^\nu}{d\tau}=-\Gamma^\nu_{\sigma\lambda}u^\sigma u^\lambda \tag{13}$$

Plug (12) and (13) into A^μ (6) yields

$$\begin{aligned}A^\mu=&-\Gamma^\nu_{\sigma\lambda}\Gamma^\mu_{\nu\rho}u^\sigma u^\lambda S^\rho+\partial_\lambda\Gamma^\mu_{\nu\rho}u^\nu u^\lambda S^\rho+\Gamma^\mu_{\nu\rho}\Gamma^\rho_{\sigma\lambda}u^\nu u^\sigma S^\lambda-\partial_\rho\Gamma^\mu_{\nu\sigma}u^\nu u^\sigma S^\rho\\=&\left(\partial_\nu\Gamma^\mu_{\sigma\rho}-\partial_\rho\Gamma^\mu_{\sigma\nu}+\Gamma^\mu_{\nu\lambda}\Gamma^\lambda_{\sigma\rho}-\Gamma^\mu_{\rho\lambda}\Gamma^\lambda_{\sigma\nu}\right)u^\nu u^\sigma S^\rho\\=&R^\mu{}_{\sigma\nu\rho}u^\nu u^\sigma S^\rho\end{aligned} \tag{14}$$

At last, we found the geodesic deviation equation

$$A^\rho=R^\rho{}_{\sigma\mu\nu}u^\sigma u^\mu S^\nu \tag{15}$$

The geodesic deviation equation

where $R^\rho{}_{\sigma\mu\nu}$ is the *Riemann curvature tensor* (its components in coordinate basis)

$$R^\rho{}_{\sigma\mu\nu}=\partial_\mu\Gamma^\rho_{\sigma\nu}-\partial_\nu\Gamma^\rho_{\sigma\mu}+\Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\sigma\nu}-\Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\sigma\mu} \tag{16}$$

Riemann curvature tensor

The relative acceleration between two neighboring geodesics is proportional to the curvature. Physically, this acceleration is interpreted as a manifestation of gravitational tidal forces.

2 Newtonian Spacetime Is Curved

Calculate the Riemann curvature tensor of a Newtonian spacetime consisting of a gravitational potential $\phi\left(x^i\right)$.

Consider a Newtonian spacetime (non relativistic). There are flat spatial metric and temporal metric, but no spacetime metric. Instead, there is an absolute time function, giving the absolute time t at any point in spacetime. We can use the absolute time as a parameter and write a trivial equation of motion for t, and the equation of motion for a particle in a Newtonian static gravitational field $\phi\left(x^i\right)$. In inertial Cartesian coordinates,

$$\frac{d^2t}{dt^2} = 0 \tag{17}$$

$$\frac{d^2x^i}{dt^2} + \partial^i\phi\frac{dt}{dt}\frac{dt}{dt} = 0 \tag{18}$$

Repackage (17) and (18) in spacetime tensorial form as

$$\frac{d^2x^\rho}{dt^2} + \Gamma^\rho_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} = 0 \tag{19}$$

where the only non-vanishing Γ's are

$$\boxed{\Gamma^i_{tt} = \partial^i\phi} \tag{20}$$

So, a particle only under the influence of gravity is moving along geodesics of spacetime. This is a manifestation of the equivalence principle, gravity is a geometry of spacetime (again - not relativistic at all).

The Riemann tensor components $R^t{}_{\sigma\mu\nu} = 0$ since

$$R^t{}_{\sigma\mu\nu} = \partial_\mu\Gamma^t_{\sigma\nu} - \partial_\nu\Gamma^t_{\sigma\mu} + \Gamma^t_{\mu\lambda}\Gamma^\lambda_{\sigma\nu} - \Gamma^t_{\nu\lambda}\Gamma^\lambda_{\sigma\mu} \tag{21}$$

and any $\Gamma^t_{\mu\nu} = 0$. The Riemann tensor components $R^i{}_{\sigma\mu\nu}$ are

$$\begin{aligned} R^i{}_{\sigma\mu\nu} &= \partial_\mu\Gamma^i_{\sigma\nu} - \partial_\nu\Gamma^i_{\sigma\mu} + \Gamma^i_{\mu\lambda}\Gamma^\lambda_{\sigma\nu} - \Gamma^i_{\nu\lambda}\Gamma^\lambda_{\sigma\mu} \\ &= \partial_\mu\Gamma^i_{\sigma\nu} - \partial_\nu\Gamma^i_{\sigma\mu} \end{aligned} \tag{22}$$

since if the index $\lambda = t$ $\Gamma^t_{\sigma\nu} = 0$ and if $\lambda = j$ then $\Gamma^i_{\mu j} = 0$. Also, we must have $\sigma = t$, and one temporal and one spatial remaining indices. The only non-vanishing components are

$$\boxed{R^i{}_{tjt} = -R^i{}_{ttj} = \partial_j\Gamma^i_{tt} = \partial_j\partial^i\phi} \tag{23}$$

Newtonian Riemann curvature tensor

We found that the Newtonian curvature tensor (23) is indeed the tidal tensor as in the Newtonian deviation equation (3). We found that while **space** is flat, the Newtonian **spacetime** is curved if there is non-homogeneous gravitational force $\partial^i\phi$. Notice also that the *Ricci tensor*

$$R_{\mu\nu} \equiv R^{\rho}{}_{\mu\rho\nu} \tag{24}$$

has one non vanishing component

Newtonian Ricci tensor

$$R_{tt} = R^{i}{}_{tit} = \partial_i\partial^i\phi = \nabla^2\phi \tag{25}$$

and by Poisson equation it equals the mass density source

$$R_{tt} = 4\pi\rho \tag{26}$$

3 Surfaces Of Constant Curvature

Consider the 2-dimensional metric of the form

$$ds^2 = a^2\left(d\chi^2 + f(\chi)^2\, d\psi^2\right) \tag{27}$$

There are three classical cases:

Hyperboloid: For $f(\chi) = \sinh(\chi)$, denote $\psi \equiv \phi$, it is the hyperboloid (hyperbolic plane) of radius a with metric

$$ds^2 = a^2\left(d\chi^2 + \sinh^2\chi d\phi^2\right) \tag{28}$$

Plane: For $f(\chi) = \chi$, denote $a\chi \equiv r$, $\psi \equiv \theta$, it is the Euclidean plane with metric

$$ds^2 = dr^2 + r^2 d\theta^2 \tag{29}$$

Sphere: For f$(\chi) = \sin\chi$, denote $\chi \equiv \theta$, $\psi \equiv \phi$, it is the sphere of radius a with metric

$$ds^2 = a^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{30}$$

Exercise: Calculate the Riemann tensor, Ricci tensor and Ricci scalar of the metric (27), and in particular of the three classical cases above (28),(29),(30). What do they have in common?

We calculated the Christoffel symbols of metric (27) in previous tutorial.

The non vanishing ones are

$$\Gamma^{\chi}_{\psi\psi} = -ff' \tag{31}$$

$$\Gamma^{\psi}_{\psi\chi} = \frac{f'}{f} \tag{32}$$

3.1 Riemann Tensor

Use formula (16). The last two indices of the Riemann tensor are antisymmetric, therefore they must have different indices, so there are four possible components to consider: ${R^{\chi}}_{\psi\chi\psi}$, ${R^{\chi}}_{\chi\chi\psi}$, ${R^{\psi}}_{\chi\chi\psi}$,${R^{\psi}}_{\psi\chi\psi}$.

$$\begin{aligned}
{R^{\chi}}_{\psi\chi\psi} = &\partial_\chi \Gamma^{\chi}_{\psi\psi} - \partial_\psi \Gamma^{\chi}_{\psi\chi} + \Gamma^{\chi}_{\chi k}\Gamma^{k}_{\psi\psi} - \Gamma^{\chi}_{\psi k}\Gamma^{k}_{\psi\chi} \\
=&\partial_\chi \Gamma^{\chi}_{\psi\psi} - \Gamma^{\chi}_{\psi\psi}\Gamma^{\psi}_{\psi\chi} \\
=&(-ff')' - (-ff')\left(\frac{f'}{f}\right) = -(f')^2 - ff'' + (f')^2 \\
=&-ff''
\end{aligned} \tag{33}$$

since derivatives with respect to ψ vanish, and the Γ does not vanish here only when the index $k = \psi$.

$${R^{\chi}}_{\chi\chi\psi} = \partial_\chi \Gamma^{\chi}_{\chi\psi} - \partial_\psi \Gamma^{\chi}_{\chi\chi} + \Gamma^{\chi}_{\chi\lambda}\Gamma^{\lambda}_{\chi\psi} - \Gamma^{\chi}_{\psi\lambda}\Gamma^{\lambda}_{\chi\chi} = 0 \tag{34}$$

since all the terms have Γ with two χ indices. Likewise,

$${R^{\psi}}_{\psi\chi\psi} = 0 \tag{35}$$

$$\begin{aligned}
{R^{\psi}}_{\chi\chi\psi} = &\partial_\chi \Gamma^{\psi}_{\chi\psi} - \partial_\psi \Gamma^{\psi}_{\chi\chi} + \Gamma^{\psi}_{\chi k}\Gamma^{k}_{\chi\psi} - \Gamma^{\psi}_{\psi k}\Gamma^{k}_{\chi\chi} \\
=&\partial_\chi \Gamma^{\psi}_{\chi\psi} + \Gamma^{\psi}_{\chi\psi}\Gamma^{\psi}_{\chi\psi} \\
=&\left(\frac{f'}{f}\right)' + \left(\frac{f'}{f}\right)^2 = \frac{f''f - (f')^2}{f^2} + \frac{(f')^2}{f^2} \\
=&\frac{f''}{f}
\end{aligned} \tag{36}$$

Collecting results (33) and (36), the non-vanishing independent Riemann tensor components are

$$R^{\chi}{}_{\psi\chi\psi} = -ff'' \tag{37}$$

$$R^{\psi}{}_{\chi\psi\chi} = -\frac{f''}{f} \tag{38}$$

Sanity check: Notice that for the Euclidean plane $f'' = 0$, so indeed all components of the Riemann tensor vanish and the space is flat. For the sphere

$$R^{\theta}{}_{\phi\theta\phi} = \sin^2\theta \tag{39}$$

$$R^{\phi}{}_{\theta\phi\theta} = 1 \tag{40}$$

3.2 Ricci Tensor

The only non-vanishing Ricci tensor components are

$$R_{\chi\chi} = R^{k}{}_{\chi k\chi} = R^{\psi}{}_{\chi\psi\chi} = -\frac{f''}{f} \tag{41}$$

$$R_{\psi\psi} = R^{k}{}_{\psi k\psi} = R^{\chi}{}_{\psi\chi\psi} = -ff'' \tag{42}$$

Again, for the plane all is zero, and for the sphere

$$R_{\theta\theta} = 1 \tag{43}$$

$$R_{\phi\phi} = \sin^2\theta \tag{44}$$

3.3 Ricci Scalar

The *Ricci scalar* is

$$R = g^{ij}R_{ij} = g^{\chi\chi}R_{\chi\chi} + g^{\psi\psi}R_{\psi\psi} = \frac{1}{a^2}\left(-\frac{f''}{f}\right) + \frac{1}{a^2}\frac{1}{f^2}\left(-ff''\right) \tag{45}$$

$$R = -\frac{2}{a^2}\frac{f''}{f} \tag{46}$$

Only when $f(\chi)$ is trigonometric, hyperbolic or linear function, does $\frac{f''}{f}$ is constant, and so is R.

Hyperboloid: $f'' = +f$. It has a constant **negative** curvature scalar

$$R_{hyperboloid} = -\frac{2}{a^2} \tag{47}$$

Plane: $f'' = 0$. It has constant **zero** curvature scalar

$$R_{plane} = 0 \tag{48}$$

Sphere: $f'' = -f$. It has a constant **positive** curvature scalar

$$R_{sphere} = \frac{2}{a^2} \tag{49}$$

a is the *curvature radius*. Notice that it does not appear in the Riemann and Ricci tensors, which are build from Γ - all spheres/hyperboloids have the same shape. The Ricci scalar is build by contraction with the metric, and accounts that the bigger the sphere/hyperboloid, the smaller (in absolute value) curvature scalar.

These are the three two-dimensional geometries with constant curvature scalar (this is a local statement). They are maximally symmetric, they consist of three linearly independent Killing vector fields.