Exercise 9 - Solution

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Gravity 1 2022-2023

Contents

1 Covariant Derivative Of Vector Fields

The first term is the partial derivative, and then for any **upper** index we **add** a Christoffel term, and any **lower** index we **subtract** a Christoffel term.

$$\nabla_q A_{jk} = \partial_q A_{jk} - \Gamma^m_{qj} A_{mk} - \Gamma^m_{qk} A_{jm} \tag{1}$$

$$\nabla_q A^{jk} = \partial_q A^{jk} + \Gamma^j_{qm} A^{mk} + \Gamma^k_{qm} A^{jm} \tag{2}$$

$$\nabla_q A^j{}_k = \partial_q A^j{}_k + \Gamma^j_{qm} A^m_k - \Gamma^m_{qk} A^j{}_m \tag{3}$$

$$\nabla_q A^j{}_{kl} = \partial_q A^j{}_{kl} + \Gamma^j_{qm} A^m_{kl} - \Gamma^m_{qk} A^j{}_{ml} - \Gamma^m_{ql} A^j{}_{km} \tag{4}$$

$$\nabla_q A^{jkl}{}_{mn} = \partial_q A^{jkl}{}_{mn} + \Gamma^j_{qs} A^{skl}{}_{mn} + \Gamma^k_{qs} A^{jsl}{}_{mn} + \Gamma^l_{qs} A^{jks}{}_{mn} - \Gamma^s_{qm} A^{jkl}{}_{sn} - \Gamma^s_{qn} A^{jkl}{}_{ms} \tag{5}$$

2 Christoffel Symbols From Metric Compatiblity And Torsion-Free Conditions

Write the three cyclic permutations of the metric compatibility condition

$$\nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma^\sigma_{\rho\mu} g_{\sigma\nu} - \Gamma^\sigma_{\rho\nu} g_{\mu\sigma} = 0 \tag{6}$$

$$\nabla_\nu g_{\rho\mu} = \partial_\nu g_{\rho\mu} - \Gamma^\sigma_{\nu\rho} g_{\sigma\mu} - \Gamma^\sigma_{\nu\mu} g_{\rho\sigma} = 0 \tag{7}$$

$$\nabla_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma^\sigma_{\mu\nu} g_{\sigma\rho} - \Gamma^\sigma_{\mu\rho} g_{\nu\sigma} = 0 \tag{8}$$

Subtract (6)−(7)−(8)

$$\partial_\rho g_{\mu\nu} - \partial_\nu g_{\rho\mu} - \partial_\mu g_{\nu\rho} + 2\Gamma^\sigma_{\nu\mu} g_{\rho\sigma} = 0 \tag{9}$$

where we used that the connection is torsion free $\Gamma^\sigma_{\mu\nu} = \Gamma^\sigma_{\nu\mu}$.

Contract with $g^{\lambda\rho}$

$$\Gamma^\sigma_{\nu\mu} g_{\rho\sigma} g^{\lambda\rho} = \Gamma^\sigma_{\nu\mu} \delta^\lambda_\sigma = \Gamma^\lambda_{\nu\mu} \tag{10}$$

so we get

$$\boxed{\Gamma^\lambda_{\nu\mu} = \frac{1}{2} g^{\lambda\rho} \left(\partial_\nu g_{\rho\mu} + \partial_\mu g_{\nu\rho} - \partial_\rho g_{\nu\mu} \right)} \tag{11}$$

3 Transformation Law For The Christoffel Symbols

3.1 From Covariant Derivative Transformation

Transformation of a $(1,1)$ tensor field

$$\nabla_{i'} V^{j'} = \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^i}{\partial x^{i'}} \nabla_i V^j \tag{12}$$

$$\frac{\partial}{\partial x^{i'}} V^{j'} + \Gamma^{j'}_{i'k'} V^{k'} = \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial}{\partial x^i} V^j + \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^i}{\partial x^{i'}} \Gamma^j_{ik} V^k \tag{13}$$

Use the chain rule for the first term of the r.h.s,

$$\frac{\partial x^i}{\partial x^{i'}} \frac{\partial}{\partial x^i} V^j = \frac{\partial}{\partial x^{i'}} V^j \tag{14}$$

(13) becomes

$$\frac{\partial}{\partial x^{i'}} V^{j'} + \Gamma^{j'}_{i'k'} V^{k'} = \frac{\partial x^{j'}}{\partial x^j} \frac{\partial}{\partial x^{i'}} V^j + \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^i}{\partial x^{i'}} \Gamma^j_{ik} V^k \tag{15}$$

Transform all vectors on the r.h.s to $V^{k'}$

$$V^j = \frac{\partial x^j}{\partial x^{k'}} V^{k'} \qquad V^k = \frac{\partial x^k}{\partial x^{k'}} V^{k'} \tag{16}$$

(15) becomes

$$\frac{\partial}{\partial x^{i'}} V^{j'} + \Gamma^{j'}_{i'k'} V^{k'} = \frac{\partial x^{j'}}{\partial x^j} \frac{\partial}{\partial x^{i'}} \left(\frac{\partial x^j}{\partial x^{k'}} V^{k'} \right) + \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^k}{\partial x^{k'}} \Gamma^j_{ik} V^{k'} \tag{17}$$

Open the first term on the r.h.s

$$\begin{aligned}
\frac{\partial x^{j'}}{\partial x^j} \frac{\partial}{\partial x^{i'}} \left(\frac{\partial x^j}{\partial x^{k'}} V^{k'} \right) &= \frac{\partial x^{j'}}{\partial x^j} \frac{\partial}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{k'}} V^{k'} + \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^j}{\partial x^{k'}} \frac{\partial}{\partial x^{i'}} V^{k'} \\
&= \frac{\partial x^{j'}}{\partial x^j} \frac{\partial}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{k'}} V^{k'} + \frac{\partial x^{j'}}{\partial x^{k'}} \frac{\partial}{\partial x^{i'}} V^{k'} \\
&= \frac{\partial x^{j'}}{\partial x^j} \frac{\partial}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{k'}} V^{k'} + \delta^{j'}_{k'} \frac{\partial}{\partial x^{i'}} V^{k'} \\
&= \frac{\partial x^{j'}}{\partial x^j} \frac{\partial}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{k'}} V^{k'} + \frac{\partial}{\partial x^{i'}} V^{j'}
\end{aligned} \tag{18}$$

(17) becomes

$$\cancel{\frac{\partial}{\partial x^{i'}} V^{j'}} + \Gamma^{j'}_{i'k'} V^{k'} = \frac{\partial x^{j'}}{\partial x^j} \frac{\partial}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{k'}} V^{k'} + \cancel{\frac{\partial}{\partial x^{i'}} V^{j'}} + \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^k}{\partial x^{k'}} \Gamma^j_{ik} V^{k'} \tag{19}$$

It is true for any $V^{k'}$, so conclude

$$\boxed{\Gamma^{j'}_{i'k'} = \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^k}{\partial x^{k'}} \Gamma^j_{ik} + \frac{\partial x^{j'}}{\partial x^l} \frac{\partial^2 x^l}{\partial x^{i'} \partial x^{k'}}} \tag{20}$$

3.2 From Metric Transformation

The Christoffel symbols are given by the metric as

$$\Gamma^{j'}_{i'k'} = \frac{1}{2} g^{j'm'} \left(\frac{\partial}{\partial x^{i'}} g_{k'm'} + \frac{\partial}{\partial x^{k'}} g_{i'm'} - \frac{\partial}{\partial x^{m'}} g_{i'k'} \right) \tag{21}$$

Transformation of $\frac{\partial}{\partial x^{i'}} g_{k'm'}$

$$\begin{aligned}
\frac{\partial}{\partial x^{i'}} g_{k'm'} &= \frac{\partial x^i}{\partial x^{i'}} \frac{\partial}{\partial x^i} \left(\frac{\partial x^k}{\partial x^{k'}} \frac{\partial x^m}{\partial x^{m'}} g_{km} \right) \\
&= \frac{\partial x^i}{\partial x^{i'}} \left(\frac{\partial}{\partial x^i} \frac{\partial x^k}{\partial x^{k'}} \right) \frac{\partial x^m}{\partial x^{m'}} g_{km} + \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^k}{\partial x^{k'}} \left(\frac{\partial}{\partial x^i} \frac{\partial x^m}{\partial x^{m'}} \right) g_{km} + \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^k}{\partial x^{k'}} \frac{\partial x^m}{\partial x^{m'}} \frac{\partial}{\partial x^i} g_{km} \\
&= \left(\frac{\partial}{\partial x^{i'}} \frac{\partial x^k}{\partial x^{k'}} \right) \frac{\partial x^m}{\partial x^{m'}} g_{km} + \frac{\partial x^k}{\partial x^{k'}} \left(\frac{\partial}{\partial x^{i'}} \frac{\partial x^m}{\partial x^{m'}} \right) g_{km} + \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^k}{\partial x^{k'}} \frac{\partial x^m}{\partial x^{m'}} \frac{\partial}{\partial x^i} g_{km}
\end{aligned} \tag{22}$$

Likewise,

$$\frac{\partial}{\partial x^{k'}} g_{i'm'} = \left(\frac{\partial}{\partial x^{k'}} \frac{\partial x^i}{\partial x^{i'}} \right) \frac{\partial x^m}{\partial x^{m'}} g_{im} + \frac{\partial x^i}{\partial x^{i'}} \left(\frac{\partial}{\partial x^{k'}} \frac{\partial x^m}{\partial x^{m'}} \right) g_{im} + \frac{\partial x^k}{\partial x^{k'}} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^m}{\partial x^{m'}} \frac{\partial}{\partial x^k} g_{im} \tag{23}$$

$$\frac{\partial}{\partial x^{m'}} g_{i'k'} = \left(\frac{\partial}{\partial x^{m'}} \frac{\partial x^i}{\partial x^{i'}} \right) \frac{\partial x^k}{\partial x^{k'}} g_{ik} + \frac{\partial x^i}{\partial x^{i'}} \left(\frac{\partial}{\partial x^{m'}} \frac{\partial x^k}{\partial x^{k'}} \right) g_{ik} + \frac{\partial x^m}{\partial x^{m'}} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^k}{\partial x^{k'}} \frac{\partial}{\partial x^m} g_{ik} \tag{24}$$

Use the same dummy indices p, q for all the expressions, and transform the inverse metric $g^{j'm'} = g^{jm} \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^{m'}}{\partial x^m}$, the Chirstoffel (21) is

$$
\begin{aligned}
\Gamma^{j'}_{i'k'} =& \\
=& \frac{1}{2} g^{jm} \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^{m'}}{\partial x^m} \left(\frac{\partial x^p}{\partial x^{m'}} \frac{\partial}{\partial x^{i'}} \frac{\partial x^q}{\partial x^{k'}} + \cancel{\frac{\partial x^p}{\partial x^{k'}} \frac{\partial}{\partial x^{i'}} \frac{\partial x^q}{\partial x^{m'}}} \right) g_{pq} \\
&+ \frac{1}{2} g^{jm} \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^{m'}}{\partial x^m} \left(\frac{\partial x^p}{\partial x^{m'}} \frac{\partial}{\partial x^{k'}} \frac{\partial x^q}{\partial x^{i'}} + \cancel{\frac{\partial x^p}{\partial x^{i'}} \frac{\partial}{\partial x^{k'}} \frac{\partial x^q}{\partial x^{m'}}} \right) g_{pq} \\
&- \frac{1}{2} g^{jm} \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^{m'}}{\partial x^m} \left(\cancel{\frac{\partial x^p}{\partial x^{k'}} \frac{\partial}{\partial x^{m'}} \frac{\partial x^q}{\partial x^{i'}}} + \cancel{\frac{\partial x^p}{\partial x^{i'}} \frac{\partial}{\partial x^{m'}} \frac{\partial x^q}{\partial x^{k'}}} \right) g_{pq} \\
&+ \frac{1}{2} g^{jn} \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^{m'}}{\partial x^n} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^k}{\partial x^{k'}} \frac{\partial x^m}{\partial x^{m'}} \left(\frac{\partial}{\partial x^i} g_{km} + \frac{\partial}{\partial x^k} g_{im} - \frac{\partial}{\partial x^m} g_{ik} \right) \\
=& g^{jm} \frac{\partial x^{j'}}{\partial x^j} \delta^p_m \frac{\partial}{\partial x^{i'}} \frac{\partial x^q}{\partial x^{k'}} g_{pq} + \frac{1}{2} g^{jn} \delta^m_n \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^k}{\partial x^{k'}} \left(\frac{\partial}{\partial x^i} g_{km} + \frac{\partial}{\partial x^k} g_{im} - \frac{\partial}{\partial x^m} g_{ik} \right) \\
=& \frac{\partial x^{j'}}{\partial x^q} \frac{\partial}{\partial x^{i'}} \frac{\partial x^q}{\partial x^{k'}} + \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^k}{\partial x^{k'}} \frac{1}{2} g^{jm} \left(\frac{\partial}{\partial x^i} g_{km} + \frac{\partial}{\partial x^k} g_{im} - \frac{\partial}{\partial x^m} g_{ik} \right) \\
=& \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^k}{\partial x^{k'}} \Gamma^j_{ik} + \frac{\partial x^{j'}}{\partial x^q} \frac{\partial^2 x^q}{\partial x^{i'} \partial x^{k'}} \qquad (25)
\end{aligned}
$$

where we used things like $\frac{\partial x^{m'}}{\partial x^n} \frac{\partial x^m}{\partial x^{m'}} = \frac{\partial x^m}{\partial x^n} = \delta^m_n$, $g^{jm} \delta^p_m g_{pq} = g^{jp} g_{pq} = \delta^j_q$, ecc., and that partial derivatives commute.

3.3 Christoffels Of Polar Coordinates

Start with Cartesian coordinates $\left(x^1 = x, x^2 = y\right)$, where all $\Gamma^j_{ik} = 0$.

By (20), we need to calculate

$$\Gamma^{j'}_{i'k'} = \frac{\partial x^{j'}}{\partial x^l} \frac{\partial^2 x^l}{\partial x^{i'} \partial x^{k'}} = \left[J^{-1} \partial_{i'} J\right]^{j'}{}_{k'} \qquad (26)$$

Polar coordinates are $\left(x^{1'} = r, x^{2'} = \theta\right)$

$$x = rcos\theta \qquad y = rsin\theta \qquad (27)$$

Compute the partial derivatives, the Jacobian matrix is:

$$J^l{}_{k'} = \frac{\partial x^l}{\partial x^{k'}} = \begin{pmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{pmatrix} \qquad (28)$$

The inverse transformation is

$$r^2 = x^2 + y^2 \qquad tan\theta = \frac{y}{x} \tag{29}$$

Compute the partial derivatives:

$$r\frac{\partial r}{\partial x} = x \;\Rightarrow\; \frac{\partial r}{\partial x} = \frac{x}{r} = cos\theta \tag{30}$$

$$r\frac{\partial r}{\partial y} = y \;\Rightarrow\; \frac{\partial r}{\partial y} = \frac{y}{r} = sin\theta \tag{31}$$

$$\frac{1}{cos^2\theta}\frac{\partial \theta}{\partial x} = -\frac{y}{x^2} \;\Rightarrow\; \frac{\partial \theta}{\partial x} = -\frac{ycos^2\theta}{x^2} = -\frac{sin\theta}{r} \tag{32}$$

$$\frac{1}{cos^2\theta}\frac{\partial \theta}{\partial y} = \frac{1}{x} \;\Rightarrow\; \frac{\partial \theta}{\partial y} = \frac{cos^2\theta}{x} = \frac{cos\theta}{r} \tag{33}$$

The inverse Jacobian matrix is

$$J^{k'}_{l} = \frac{\partial x^{k'}}{\partial x^{l}} = \begin{pmatrix} cos\theta & sin\theta \\ -\frac{sin\theta}{r} & \frac{cos\theta}{r} \end{pmatrix} \tag{34}$$

Calculate Christoffels $\Gamma^{j'}_{rk'}$ by plugin (28) and (34) into (26) with $i' = r$

$$\begin{aligned} \Gamma^{j'}_{rk'} &= \begin{pmatrix} cos\theta & sin\theta \\ -\frac{sin\theta}{r} & \frac{cos\theta}{r} \end{pmatrix} \frac{\partial}{\partial r} \begin{pmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{pmatrix} \\ &= \begin{pmatrix} cos\theta & sin\theta \\ -\frac{sin\theta}{r} & \frac{cos\theta}{r} \end{pmatrix} \begin{pmatrix} 0 & -sin\theta \\ 0 & cos\theta \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{sin^2\theta}{r} + \frac{cos^2\theta}{r} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{r} \end{pmatrix} \end{aligned} \tag{35}$$

The non-vanishing Christoffel here is

$$\boxed{\Gamma^{\theta}_{r\theta} = \frac{1}{r}} \tag{36}$$

Calculate Christoffels $\Gamma^{j'}_{\theta k'}$ by plugin (28) and (34) into (26) with $i' = \theta$

$$\begin{aligned} \Gamma^{j'}_{\theta k'} &= \begin{pmatrix} cos\theta & sin\theta \\ -\frac{sin\theta}{r} & \frac{cos\theta}{r} \end{pmatrix} \frac{\partial}{\partial \theta} \begin{pmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{pmatrix} \\ &= \begin{pmatrix} cos\theta & sin\theta \\ -\frac{sin\theta}{r} & \frac{cos\theta}{r} \end{pmatrix} \begin{pmatrix} -sin\theta & -rcos\theta \\ cos\theta & -rsin\theta \end{pmatrix} = \begin{pmatrix} 0 & -r \\ \frac{1}{r} & 0 \end{pmatrix} \end{aligned}$$

The remaining non-vanishing Christoffels are

$$\Gamma^{\theta}_{\theta r} = \frac{1}{r} \qquad \Gamma^{r}_{\theta\theta} = -r \tag{37}$$

You can also calculate each component separately, without use of matrices.

4 Laplacian

4.1 Formula

See recitation 9 section 2.3.

$$\nabla^2 f = \frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g} g^{ij} \partial_j f \right) \tag{38}$$

4.2 Cylindrical Coordinates

The metric in cylindrical coordinates (ρ, θ, z) is

$$ds^2 = d\rho^2 + \rho^2 d\theta^2 + dz^2 \tag{39}$$

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\rho^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{40}$$

$$g = det\left(g_{ij}\right) = \rho^2 \tag{41}$$

$$\sqrt{g} = \rho \tag{42}$$

Notice that the metric is diagonal.

$$\nabla^2 f = \frac{1}{\rho} \partial_\rho \left(\rho \partial_\rho f \right) + \frac{1}{\rho} \partial_\theta \left(\rho \frac{1}{\rho^2} \partial_\theta f \right) + \frac{1}{\rho} \partial_z \left(\rho \partial_z f \right) \tag{43}$$

$$\nabla^2 f = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial f}{\partial z^2} \tag{44}$$

4.3 Spherical Coordinates

The metric in spherical coordinates (r, θ, ϕ) is

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2 \tag{45}$$

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 sin^2\theta \end{pmatrix} \qquad g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2 sin^2\theta} \end{pmatrix} \tag{46}$$

$$g = det\left(g_{ij}\right) = r^4 sin^2\theta \tag{47}$$

$$\sqrt{g} = r^2 sin\theta \tag{48}$$

Notice that the metric is diagonal.

The short way: Use metric determinant and inverse metric

$$\nabla^2 f = \frac{1}{r^2 sin\theta}\partial_r\left(r^2 sin\theta \partial_r f\right) + \frac{1}{r^2 sin\theta}\partial_\theta\left(r^2 sin\theta \frac{1}{r^2}\partial_\theta f\right) + \frac{1}{r^2 sin\theta}\partial_\phi\left(r^2 sin\theta \frac{1}{r^2 sin^2\theta}\partial_\phi f\right) \tag{49}$$

$$\boxed{\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}} \tag{50}$$

The long and winding road:

Use Christoffels (extra work that is skipped here)

$$\Gamma^r_{\theta\theta} = -r \qquad \Gamma^\theta_{\theta r} = \frac{1}{r} \qquad \Gamma^r_{\phi\phi} = -r sin^2\theta \qquad \Gamma^\phi_{\phi r} = \frac{1}{r} \tag{51}$$

$$\Gamma^\theta_{\phi\phi} = -sin\theta cos\theta \qquad \Gamma^\phi_{\phi\theta} = cot\theta \tag{52}$$

$$\begin{aligned} g^{ij}\nabla_i\nabla_j f = {} & \nabla_r\partial_r f + \frac{1}{r^2}\nabla_\theta\partial_\theta f + \frac{1}{r^2 sin^2\theta}\nabla_\phi\partial_\phi f \\ = {} & \partial_r\partial_r f - \Gamma^k_{rr}\partial_k f + \frac{1}{r^2}\left(\partial_\theta\partial_\theta f - \Gamma^k_{\theta\theta}\partial_k f\right) + \frac{1}{r^2 sin^2\theta}\left(\partial_\phi\partial_\phi f - \Gamma^k_{\phi\phi}\partial_k f\right) \\ = {} & \partial_r\partial_r f + \frac{1}{r^2}\partial_\theta\partial_\theta f - \frac{1}{r^2}\Gamma^r_{\theta\theta}\partial_r f + \frac{1}{r^2 sin^2\theta}\partial_\phi\partial_\phi f - \frac{1}{r^2 sin^2\theta}\Gamma^r_{\phi\phi}\partial_r f - \frac{1}{r^2 sin^2\theta}\Gamma^\theta_{\phi\phi}\partial_\theta f \\ = {} & \partial_r\partial_r f + \frac{1}{r^2}\partial_\theta\partial_\theta f + \frac{1}{r}\partial_r f + \frac{1}{r^2 sin^2\theta}\partial_\phi\partial_\phi f + \frac{1}{r}\partial_r f + \frac{1}{r^2 sin\theta}cos\theta\partial_\theta f \\ = {} & \partial_r\partial_r f + \frac{2}{r}\partial_r f + \frac{1}{r^2}\partial_\theta\partial_\theta f + \frac{1}{r^2 sin\theta}\left(\partial_\theta sin\theta\right)\partial_\theta f + \frac{1}{r^2 sin^2\theta}\partial_\phi\partial_\phi f \\ = {} & \partial_r\partial_r f + \frac{1}{r^2}\left(\partial_r r^2\right)\partial_r f + \frac{1}{r^2}\partial_\theta\partial_\theta f + \frac{1}{r^2 sin\theta}\left(\partial_\theta sin\theta\right)\partial_\theta f + \frac{1}{r^2 sin^2\theta}\partial_\phi\partial_\phi f \\ = {} & \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} \end{aligned} \tag{53}$$

We got the same result!

5 Killing Equation For The Euclidean Plane

5.1 In Cartesian Coordinates

In Cartesian coordinates $g_{ij} = \delta_{ij}$ and all $\Gamma^i_{jk} = 0$. So there is no difference between upper and lower indices, and covariant derivatives reduce to partial derivatives. Therefore, Killing equation takes the form

$$\partial_i \xi^j + \partial_j \xi^i = 0 \tag{54}$$

Clearly for both

$$X^i = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad Y^i = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{55}$$

$$\begin{bmatrix} 2\partial_x \xi^x & \partial_x \xi^y + \partial_y \xi^x \\ \partial_y \xi^x + \partial_x \xi^y & 2\partial_y \xi^y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{56}$$

For

$$R^i = \begin{pmatrix} -y \\ x \end{pmatrix} \tag{57}$$

$$\begin{bmatrix} 2\partial_x R^x & \partial_x R^y + \partial_y R^x \\ " & 2\partial_y R^y \end{bmatrix} = \begin{bmatrix} 0 & 1-1 \\ " & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{58}$$

Remark: xy and yx components are identical so there is no need to calculate them both, we wrote " to indicate it is the same as the other component.

5.2 Transform To Polar Coordinates

We already found the matrices (28)

$$\frac{\partial x^i}{\partial x^{k'}} = \begin{pmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{pmatrix} \tag{59}$$

and (34)

$$\frac{\partial x^{k'}}{\partial x^j} = \begin{pmatrix} cos\theta & sin\theta \\ -\frac{sin\theta}{r} & \frac{cos\theta}{r} \end{pmatrix} \tag{60}$$

Multiply

$$\frac{\partial x^i}{\partial x^{k'}}\frac{\partial x^{k'}}{\partial x^j} = \begin{pmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{pmatrix} \begin{pmatrix} cos\theta & sin\theta \\ -\frac{sin\theta}{r} & \frac{cos\theta}{r} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \delta^i_j \tag{61}$$

Vector transformation:

$$X^{i'} = \frac{\partial x^{i'}}{\partial x^i} X^i = \begin{pmatrix} cos\theta & sin\theta \\ -\frac{sin\theta}{r} & \frac{cos\theta}{r} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} cos\theta \\ -\frac{sin\theta}{r} \end{pmatrix} \tag{62}$$

$$Y^{i'} = \frac{\partial x^{i'}}{\partial x^i} Y^i = \begin{pmatrix} cos\theta & sin\theta \\ -\frac{sin\theta}{r} & \frac{cos\theta}{r} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} sin\theta \\ \frac{cos\theta}{r} \end{pmatrix} \tag{63}$$

$$R^{i'} = \frac{\partial x^{i'}}{\partial x^i} R^i = \begin{pmatrix} cos\theta & sin\theta \\ -\frac{sin\theta}{r} & \frac{cos\theta}{r} \end{pmatrix} \begin{pmatrix} -rsin\theta \\ rcos\theta \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{64}$$

5.3 In Polar Coordinates

To write Killing equation in polar coordinates we need to lower the indices of the vector fields, and use the covariant derivatives. The metric is

$$g_{i'j'} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \tag{65}$$

$$X_{i'} = g_{i'j'} X^{j'} = \begin{pmatrix} X^r & r^2 X^\theta \end{pmatrix} = \begin{pmatrix} cos\theta & -rsin\theta \end{pmatrix} \tag{66}$$

$$Y_{i'} = g_{i'j'} Y^{j'} = \begin{pmatrix} Y^r & r^2 Y^\theta \end{pmatrix} = \begin{pmatrix} sin\theta & rcos\theta \end{pmatrix} \tag{67}$$

$$R_{i'} = g_{i'j'} R^{j'} = \begin{pmatrix} R^r & r^2 R^\theta \end{pmatrix} = \begin{pmatrix} 0 & r^2 \end{pmatrix} \tag{68}$$

We already calculated the Christoffels in (36) and (37).
Killing equation is

$$\nabla_{i'}\xi_{j'} + \nabla_{j'}\xi_{i'} = 0 \tag{69}$$

Its components are

$$0 = \nabla_r \xi_r = \partial_r \xi_r - \Gamma^{k'}_{rr}\xi_{k'} = \partial_r \xi_r \tag{70}$$

$$0 = \nabla_\theta \xi_\theta = \partial_\theta \xi_\theta - \Gamma^{k'}_{\theta\theta}\xi_{k'} = \partial_\theta \xi_\theta - \Gamma^r_{\theta\theta}\xi_r = \partial_\theta \xi_\theta + r\xi_r \tag{71}$$

$$0=\nabla_r\xi_\theta+\nabla_\theta\xi_r=\partial_r\xi_\theta-\Gamma^{k'}_{r\theta}\xi_{k'}+\partial_\theta\xi_r-\Gamma^{k'}_{\theta r}\xi_{k'}=\partial_r\xi_\theta-\Gamma^{k'}_{r\theta}\xi_{k'}+\partial_\theta\xi_r-\Gamma^{k'}_{\theta r}\xi_{k'}$$

$$=\partial_r\xi_\theta-\Gamma^{\theta}_{r\theta}\xi_\theta+\partial_\theta\xi_r-\Gamma^{\theta}_{\theta r}\xi_\theta=\partial_r\xi_\theta+\partial_\theta\xi_r-\frac{2}{r}\xi_\theta \tag{72}$$

Check for $X_{i'}=\begin{pmatrix}cos\theta & -rsin\theta\end{pmatrix}$:

$$\partial_r X_r=0 \tag{73}$$

$$\partial_\theta X_\theta+rX_r=-rcos\theta+rcos\theta=0 \tag{74}$$

$$\partial_r X_\theta+\partial_\theta X_r-\frac{2}{r}X_\theta=-sin\theta-sin\theta-\frac{2}{r}\left(-rsin\theta\right)=0 \tag{75}$$

Check for $Y_{i'}=\begin{pmatrix}sin\theta & rcos\theta\end{pmatrix}$:

$$\partial_r Y_r=0 \tag{76}$$

$$\partial_\theta Y_\theta+rY_r=-rsin\theta+rsin\theta=0 \tag{77}$$

$$\partial_r Y_\theta+\partial_\theta Y_r-\frac{2}{r}Y_\theta=cos\theta+cos\theta-\frac{2}{r}\left(rcos\theta\right)=0 \tag{78}$$

Check for $R_{i'}=\begin{pmatrix}0 & r^2\end{pmatrix}$:

$$\partial_r R_r=0 \tag{79}$$

$$\partial_\theta R_\theta+rR_r=0+0=0 \tag{80}$$

$$\partial_r R_\theta+\partial_\theta R_r-\frac{2}{r}R_\theta=2r+0-\frac{2}{r}\left(r^2\right)=0 \tag{81}$$

X is the Killing vector field that generates the translation symmetry in the x-direction, Y is the Killing vector field that generates the translation symmetry in the y-direction, and R is the Killing vector field that generates the rotation symmetry about the plane origin. These are three linearly independent Killing vector fields, which is the maximal number of isometries for a 2-dimensional space, i.e., the Euclidean plane is a maximally symmetric space.

6 Killing Vector Fields of Poincare Half Plane

The metric of Poincare half-plane is

$$ds^2=\frac{a^2}{y^2}\left(dx^2+dy^2\right) \tag{82}$$

$$g_{xx} = g_{yy} = \frac{a^2}{y^2} \quad g_{xy} = 0 \tag{83}$$

The Christoffel symbols are

$$\Gamma^x_{xy} = \Gamma^x_{yx} = \Gamma^y_{yy} = -\frac{1}{y} \qquad\qquad \Gamma^y_{xx} = \frac{1}{y} \tag{84}$$

Killing's equation is

$$\nabla_i \xi_j + \nabla_j \xi_i = 0 \tag{85}$$

The xx equation is

$$\begin{aligned} 0 &= \nabla_x \xi_x = \partial_x \xi_x - \Gamma^i_{xx} \xi_i = \partial_x \left(g_{xx}\xi^x\right) - \Gamma^y_{xx}\xi_y = \left(\partial_x g_{xx}\right)\xi^x + g_{xx}\partial_x \xi^x - \Gamma^y_{xx} g_{yy} \xi^y \\ &= g_{xx}\partial_x \xi^x - \Gamma^y_{xx} g_{yy}\xi^y = \frac{a^2}{y^2}\left(\partial_x \xi^x - \frac{1}{y}\xi^y\right) \end{aligned} \tag{86}$$

The yy equation is

$$\begin{aligned} 0 &= \nabla_y \xi_y = \partial_y \xi_y - \Gamma^i_{yy}\xi_i = \partial_y \left(g_{yy}\xi^y\right) - \Gamma^y_{yy}\xi_y \\ &= \left(\partial_y g_{yy}\right)\xi^y + g_{yy}\partial_y \xi^y - \Gamma^y_{yy} g_{yy}\xi^y \\ &= -\frac{2a^2}{y^3}\xi^y + \frac{a^2}{y^2}\partial_y \xi^y + \frac{1}{y}\frac{a^2}{y^2}\xi^y \\ &= \frac{a^2}{y^3}\left(y\partial_y \xi^y - \xi^y\right) \end{aligned} \tag{87}$$

The xy equation is

$$\begin{aligned} 0 &= \nabla_x \xi_y + \nabla_y \xi_x = \partial_x \xi_y + \partial_y \xi_x - 2\Gamma^i_{xy}\xi_i \\ &= \partial_x \left(g_{yy}\xi^y\right) + \partial_y \left(g_{xx}\xi^x\right) - 2\Gamma^x_{xy}\xi_x \\ &= \left(\partial_x g_{yy}\right)\xi^y + g_{yy}\partial_x \xi^y + \left(\partial_y g_{xx}\right)\xi^x + g_{xx}\partial_y \xi^x - 2\Gamma^x_{xy} g_{xx}\xi^x \\ &= \frac{a^2}{y^2}\partial_x \xi^y - \frac{2a^2}{y^3}\xi^x + \frac{a^2}{y^2}\partial_y \xi^x + 2\frac{1}{y}\frac{a^2}{y^2}\xi^x \\ &= \frac{a^2}{y^2}\left(\partial_x \xi^y + \partial_y \xi^x\right) \end{aligned}$$

Summary

$$y\partial_x \xi^x - \xi^y = 0 \tag{88}$$

$$y\partial_y \xi^y - \xi^y = 0 \tag{89}$$

$$\partial_x \xi^y + \partial_y \xi^x = 0 \tag{90}$$

From (89) ξ^y is a degree one homogeneous function of y

$$\xi^y = f(x)\, y \tag{91}$$

plug in (88) yields

$$\partial_x \xi^x = f(x) \tag{92}$$

Differentiate (90) with respect to x and plug in (91) and (92)

$$\partial_x^2 \xi^y + \partial_y \partial_x \xi^x = \partial_x^2 (f(x))\, y + \partial_y f(x) = \partial_x^2 (f(x))\, y = 0 \tag{93}$$

thus

$$\partial_x^2 f(x) = 0 \tag{94}$$

$$f(x) = A + Bx \tag{95}$$

plug back into (92)

$$\partial_x \xi^x = A + Bx \quad \Rightarrow \quad \xi^x = Ax + \frac{1}{2} B x^2 + h(y) \tag{96}$$

and then into (90)

$$\partial_y \xi^x = -By \quad \Rightarrow \quad \partial_y h(y) = -By \quad \Rightarrow \quad h(y) = -B\left(\frac{1}{2} y^2 + C\right) \tag{97}$$

putting together

$$\xi^x = Ax + \frac{B}{2}\left(x^2 - y^2\right) + D \tag{98}$$

$$\xi^y = Ay + Bxy \tag{99}$$

The Killing vector fields are of the form

$$\xi^i = \begin{pmatrix} Ax + \frac{B}{2}\left(x^2 - y^2\right) + D \\ Ay + Bxy \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} + \frac{B}{2} \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix} + D \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{100}$$

The basis Killing vector fields are

$$X^i = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad D^i = \begin{pmatrix} x \\ y \end{pmatrix} \qquad K^i = \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix} \tag{101}$$

The isometry algebra is 3-dimensional, thus the hyperbolic plane is a maximally symmetric 2-dimensional space.