Gravity 1 - Solution 6

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Contents

1 Lorentz-Poincare Half-Plane

$$ds^2 = \frac{1}{t^2}\left(-dt^2 + dx^2\right) \tag{1}$$

$t > 0,\ -\infty < x < \infty.$

1.1 Infinite Proper Time

For $dx = 0$, $d\tau = \frac{dt}{t}$,

$$\int_{t_0}^{0} d\tau = \int_{t_0}^{0} \frac{dt}{t} = \ln 0 - \ln t_0 \to \infty \tag{2}$$

1.2 The Geodesic Equations

Calculate the Christoffel symbols

$$\Gamma^x_{xx} = \frac{1}{2} g^{xx} \partial_x g_{xx} = 0 \tag{3}$$

$$\Gamma^x_{xt} = \frac{1}{2} g^{xx}\left(\partial_x g_{tx} + \partial_t g_{xx} - \partial_x g_{xt}\right) = \frac{1}{2} t^2 \left(-\frac{2}{t^3}\right) = -\frac{1}{t} \tag{4}$$

$$\Gamma^t_{xx} = \frac{1}{2} g^{tt}\left(\partial_x g_{xt} + \partial_x g_{xt} - \partial_t g_{xx}\right) = \frac{1}{2}\left(-t^2\right)\left(- - \frac{2}{t^3}\right) = -\frac{1}{t} \tag{5}$$

$$\Gamma^x_{tt} = \frac{1}{2} g^{xx}\left(\partial_t g_{tx} + \partial_t g_{tx} - \partial_x g_{tt}\right) = 0 \tag{6}$$

$$\Gamma^t_{tx} = \frac{1}{2} g^{tt}\left(\partial_t g_{xt} + \partial_x g_{tt} - \partial_t g_{tx}\right) = 0 \tag{7}$$

$$\Gamma^t_{tt} = \frac{1}{2} g^{tt} \partial_t g_{tt} = \frac{1}{2}\left(-t^2\right)\frac{2}{t^3} = -\frac{1}{t} \tag{8}$$

The geodesic equations are

$$\frac{d^2 t}{d\tau^2} + \Gamma^t_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\mu}{d\tau} = \frac{d^2 t}{d\tau^2} + \Gamma^t_{tt}\left(\frac{dt}{d\tau}\right)^2 + \Gamma^t_{xx}\left(\frac{dx}{d\tau}\right)^2 = 0 \tag{9}$$

$$\frac{d^2 x}{d\tau^2} + \Gamma^x_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\mu}{d\tau} = \frac{d^2 x}{d\tau^2} + \Gamma^x_{xt}\frac{dt}{d\tau}\frac{dx}{d\tau} = 0 \tag{10}$$

$$\frac{d^2t}{d\tau^2} - \frac{1}{t}\left(\left(\frac{dt}{d\tau}\right)^2 + \left(\frac{dx}{d\tau}\right)^2\right) = 0 \quad (11)$$

$$\frac{d^2x}{d\tau^2} - \frac{1}{t}\frac{dt}{d\tau}\frac{dx}{d\tau} = 0 \quad (12)$$

The geodesic equations

1.3 Integral of Motion

The metric is independent of the x coordinate, so there is an integral of motion

$$p = g_{x\mu}\frac{dx^\mu}{d\tau} = g_{xx}\frac{dx}{d\tau} = \frac{1}{t^2}\frac{dx}{d\tau} \quad (13)$$

1.4 Timelike Geodesics

Normalization of velocity

$$-1 = u \cdot u = g_{\mu\nu}u^\mu u^\nu = g_{tt}\left(u^t\right)^2 + g_{xx}\left(u^x\right)^2 \quad (14)$$

$$-1 = -\frac{1}{t^2}\left(u^t\right)^2 + \frac{1}{t^2}\left(pt^2\right)^2 \quad (15)$$

find the velocities

$$u^t = \frac{dt}{d\tau} = \pm t\sqrt{1 + p^2t^2} \quad (16)$$

$$u^x = \frac{dx}{d\tau} = pt^2 \quad (17)$$

divide them

$$\frac{dx}{dt} = \frac{u^x}{u^t} = \frac{pt}{\sqrt{1 + p^2t^2}} \quad (18)$$

solve the integral

$$x = \int \frac{pt}{\sqrt{1 + p^2t^2}}dt = \frac{1}{2p}\int \frac{dw}{\sqrt{w}} = \frac{1}{p}\sqrt{w} + x_0 = \frac{1}{p}\sqrt{1 + p^2t^2} + x_0 \quad (19)$$

with change of variable

$$w = 1 + p^2t^2 \quad , \quad dw = 2p^2tdt \quad (20)$$

therefore

$$(x - x_0)^2 = \frac{1}{p^2}\left(1 + p^2t^2\right) \quad (21)$$

$$-t^2 + (x - x_0)^2 = \frac{1}{p^2} \tag{22}$$

Half hyperbolas timelike geodesics

For $p = 0$

$$\frac{dx}{dt} = 0 \Rightarrow x = x_0 \tag{23}$$

Vertical lines timelike geodesics

1.5 Null Geodesics

Normalization of velocity

$$0 = u \cdot u = g_{\mu\nu} u^\mu u^\nu = g_{tt} \left(u^t\right)^2 + g_{xx} \left(u^x\right)^2 \tag{24}$$

$$0 = -\frac{1}{t^2} \left(u^t\right)^2 + \frac{1}{t^2} \left(pt^2\right)^2 \tag{25}$$

find the velocities

$$u^t = \frac{dt}{d\tau} = \pm pt^2 \tag{26}$$

$$u^x = \frac{dx}{d\tau} = pt^2 \tag{27}$$

divide them

$$\frac{dx}{dt} = \frac{u^x}{u^t} = \pm 1 \tag{28}$$

$$x = \pm t + x_0 \tag{29}$$

Null geodesics

Equivalent way, set the line element to zero $ds^2 = 0 \Rightarrow dx = \pm dt$.

2 Perturbation to Minkowski Spacetime

Given a metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{30}$$

with $h_{\mu\nu} \ll 1$, $\partial_\rho h_{\mu\nu} \ll 1$, and $v^2 \ll 1$.

$$h_{00} = -2\phi \tag{31}$$

$$h_{0i} = A_i \tag{32}$$

The leading order of the Christofell symbols is first order in h. They are

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\eta^{\rho\sigma}\left(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}\right) \tag{33}$$

We will use the relation

$$\frac{dx^i}{d\tau} = \frac{dt}{d\tau}\frac{dx^i}{dt} = \left(\frac{dt}{d\tau}\right)v^i \tag{34}$$

and

$$\frac{d^2x^i}{d\tau^2} = \frac{d}{d\tau}\left(\frac{dt}{d\tau}\frac{dx^i}{dt}\right) = \frac{d^2t}{d\tau^2}\frac{dx^i}{dt} + \frac{dt}{d\tau}\frac{d}{d\tau}\frac{dx^i}{dt} = \frac{d^2t}{d\tau^2}v^i + \left(\frac{dt}{d\tau}\right)^2\frac{d^2x^i}{dt^2} \tag{35}$$

The spatial geodesic equation is

$$\begin{aligned}\frac{d^2x^i}{d\tau^2} &= -\Gamma^i_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\\ &= -\Gamma^i_{tt}\frac{dt}{d\tau}\frac{dt}{d\tau} - 2\Gamma^i_{jt}\frac{dt}{d\tau}\frac{dx^j}{d\tau} - \Gamma^i_{jk}\frac{dx^j}{d\tau}\frac{dx^k}{d\tau}\\ &= -\Gamma^i_{tt}\left(\frac{dt}{d\tau}\right)^2 - 2\Gamma^i_{jt}\left(\frac{dt}{d\tau}\right)^2 v^j - \Gamma^i_{jk}\left(\frac{dt}{d\tau}\right)^2 v^j v^k\end{aligned} \tag{36}$$

together with the l.h.s (35)

$$\frac{d^2t}{d\tau^2}v^i + \left(\frac{dt}{d\tau}\right)^2\frac{d^2x^i}{dt^2} = -\Gamma^i_{tt}\left(\frac{dt}{d\tau}\right)^2 - 2\Gamma^i_{jt}\left(\frac{dt}{d\tau}\right)^2 v^j - \Gamma^i_{jk}\left(\frac{dt}{d\tau}\right)^2 v^j v^k \tag{37}$$

Now, since the geodesic equation for t is

$$\frac{d^2t}{d\tau^2} = -\Gamma^t_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \tag{38}$$

$\frac{d^2t}{d\tau^2}$ is at least at first order in h. Dropping all terms in (37) that are above a total of first order in h and v yields

$$\left(\frac{dt}{d\tau}\right)^2\frac{d^2x^i}{dt^2} = -\Gamma^i_{tt}\left(\frac{dt}{d\tau}\right)^2 \tag{39}$$

thus

$$\frac{d^2x^i}{dt^2} = -\Gamma^i_{tt} \tag{40}$$

We only need to calculate

$$\Gamma^i_{tt} = \frac{1}{2}\eta^{i\sigma}\left(\partial_t h_{t\sigma} + \partial_t h_{t\sigma} - \partial_\sigma h_{tt}\right) = \frac{1}{2}\eta^{ii}\left(2\partial_t h_{ti} - \partial_i h_{tt}\right)$$
$$= \frac{1}{2}\left(2\partial_t A^i - \partial_i\left(-2\phi\right)\right) = \frac{\partial A^i}{\partial t} + \frac{\partial \phi}{\partial x_i} \tag{41}$$

therefore

$$\frac{d^2x^i}{dt^2} = E^i \tag{42}$$

with

$$E^i \equiv -\frac{\partial \phi}{\partial x_i} - \frac{\partial A^i}{\partial t} \tag{43}$$

3 Surface and Normal

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3. משטח ונורמל

(a) A flat geometry with azimuthal symmetry is $ds^2 = dr^2 + dz^2 + r^2 d\phi^2$ and so we need to find a function $z = z(r)$ which will give this surface:

$$ds^2 = dr^2 \left(1 + \left(\frac{dz}{dr}\right)^2\right) + r^2 d\phi^2.$$

The equation is:

$$\begin{aligned} 1 + z'^2 &= \frac{1}{1 - 2M/r} \Rightarrow z'^2 = \frac{2M}{r - 2M} \\ \Rightarrow z &= \int \sqrt{\frac{2M}{r - 2M}} dr = s\sqrt{2M(r - 2M)} \end{aligned}$$

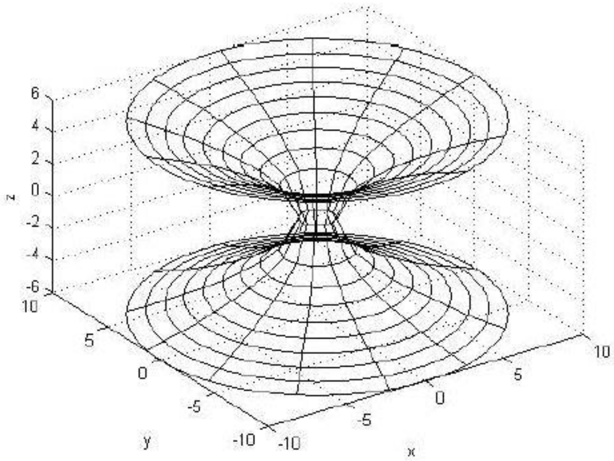

Figure 2:

(b) We can write the equation for the surface as $f(x^\alpha) = z - 2\sqrt{2M(r - 2M)} = 0$. Then the normal vector will be:

$$\begin{aligned} N_\mu &= f_{,\mu} \\ N_z &= 1 \\ N_r &= -\sqrt{\frac{2M}{r - 2M}} \\ N_\phi &= 0 \end{aligned}$$

The vector must be normalized:

$$\begin{aligned} g^{\alpha\beta} N_\alpha N_\beta &= N_z^2 + N_R^2 = 1 + \frac{2M}{r - 2M} = \frac{r}{r - 2M} \\ n_\mu &= \frac{N_\mu}{\sqrt{g^{\alpha\beta} N_\alpha N_\beta}}, \\ \vec{n} &= \left(\sqrt{1 - \frac{2M}{r}}, -\sqrt{\frac{2M}{r}}, 0\right) \end{aligned}$$