כבידה 1

2022/23 סמסטר א'

<u>פתרון תרגיל 7 – מדידות בגיאומטריית שוורצשילד</u>

1. צופים בחלקיק נופל רדיאלית

ראו תרגול 7 פרק 2.1.

2. כוח של חללית מחוץ לכוכב

Example 20.8. The Acceleration of a Stationary Observer in the Schwarzschild Geometry. In an inertial frame of special relativity or a local inertial frame in general relativity, the acceleration four-vector of a particle can be defined by its coordinate basis components as

$$a^\alpha = \frac{du^\alpha}{d\tau} \qquad \text{(LIF only)}, \tag{20.56}$$

where $\mathbf{u}$ is the particle's four-velocity and τ is the proper time along its world line. But, even in flat space, (20.56) is not correct in a general coordinate system. The correct and general definition of acceleration employs the correct and general way to differentiate a vector—the covariant derivative. Specifically, acceleration is defined generally by

$$\mathbf{a} \equiv \boldsymbol{\nabla}_\mathbf{u}\mathbf{u}. \tag{20.57}$$

This reduces to (20.56) in a local inertial frame because there $\boldsymbol{\nabla}_\mathbf{u} = u^\alpha \nabla_\alpha = u^\alpha(\partial/\partial x^\alpha) = (dx^\alpha/d\tau)(\partial/\partial x^\alpha) = d/d\tau$.

A stationary observer who remains at a fixed value of (r, θ, ϕ) in the spacetime of a Schwarzschild black hole is accelerating. Rocket thrust is needed to maintain a fixed position in space; the alternative is falling into the black hole.[7] To illustrate how to use the covariant derivative, we will calculate the components of the acceleration of a stationary observer, as defined by (20.57).

The Schwarzschild coordinate components of the normalized four-velocity of a stationary observer at radius r are [cf. (9.16)]:

$$u^\alpha = (u^t, \vec{0}) = [(1-2M/r)^{-1/2}, 0, 0, 0]. \tag{20.58}$$

Using (20.54) to evaluate (20.57), we have

$$a^\alpha \equiv u^\beta \nabla_\beta u^\alpha = u^t \nabla_t u^\alpha = u^t \left(\frac{\partial u^\alpha}{\partial t} + \Gamma^\alpha_{t\gamma} u^\gamma \right) = u^t \left(\frac{\partial u^\alpha}{\partial t} + \Gamma^\alpha_{tt} u^t \right) \tag{20.59}$$

since $\mathbf{u}$ has only a t component. Since the components u^α are independent of time, this reduces to

$$a^\alpha = \Gamma^\alpha_{tt} (u^t)^2. \tag{20.60}$$

Appendix B shows that the only nonvanishing Christoffel symbol that enters into (20.60) is $\Gamma^r_{tt} = (1 - 2M/r)(M/r^2)$. Thus,

$$a^\alpha = (0, M/r^2, 0, 0). \tag{20.61}$$

The acceleration points in the radial direction—the direction of the force necessary to keep the particle from falling into the black hole. Its components are finite at the horizon $r = 2M$, but the true measure of finiteness is its length,

$$(\mathbf{a} \cdot \mathbf{a})^{1/2} = \left(1 - \frac{2M}{r}\right)^{-1/2} \frac{M}{r^2}, \tag{20.62}$$

which diverges at $r = 2M$. Infinite acceleration is required to remain stationary at the horizon of a black hole. (See also the discussion in Box 12.2 on p. 261.)

3. זמן מחזור של מסלול מעגלי

a) From (7.48) $\Omega = d\phi/dt = (M/r^3)^{1/2}$. A clock at infinity measures t, so that Ω is the angular velocity as measured from infinity. The period measured at infinity is

$$P_\infty = \frac{2\pi}{\Omega} = 2\pi \left(\frac{r^3}{M}\right)^{\frac{1}{2}} = 2\pi 7^{\frac{3}{2}} M = 116.M$$

for $r = 7M$.

b) To get the period as measured on the spaceship we need

$$\frac{d\phi}{d\tau} = \frac{d\phi}{dt}\,\frac{dt}{d\tau} = \Omega\,\frac{dt}{d\tau}\ .$$

To find $dt/d\tau$ note that for a circular orbit in the equatorial plane

$$\begin{aligned}
\mathbf{u}\cdot\mathbf{u} = -1 &= -\left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2 \\
-1 &= -\left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + r^2\Omega^2\left(\frac{dt}{d\tau}\right)^2 \\
-1 &= -\left(1 - \frac{2M}{r} - r^2\Omega^2\right)\left(\frac{dt}{dr}\right)^2 \\
&= -\left(1 - \frac{3M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 = -\frac{4}{7}\left(\frac{dt}{dr}\right)^2\ .
\end{aligned}$$

Thus $dt/d\tau = (7/4)^{1/2}$ and

$$\frac{d\phi}{d\tau} = \left(\frac{7}{4}\right)^{\frac{1}{2}}\Omega$$

and the period $P_{\text{spaceship}} = (4/7)^{1/2} P_\infty \approx 88\,M$.

4. מסלול לא יציב ברדיוס מינימלי

a) The maximum of the effective potential occurs at a radius of [*cf.* (9.34)]

$$r_{\max} = \frac{\ell^2}{2M}\left[1-\left(1-12\left(\frac{M}{\ell}\right)^2\right)^{\frac{1}{2}}\right] .$$

This gets smaller for large ℓ and in the limit of infinite ℓ is

$$r_{\max} = \frac{\ell^2}{2M}\left[1-1+6\left(\frac{M}{\ell}\right)^2+\cdots\right] = 3M$$

which is the minimum possible radius.

To calculate the speed measured by a stationary observer we need to project the four-velocity of the particle $\mathbf{u}$ on the orthonormal basis of the observer. The components of orthonormal basis vectors pointing in the t and ϕ directions (all that will be needed) are

$$\begin{aligned} \mathbf{e}_{\hat{t}} &= \left[\left(1-\frac{2M}{r}\right)^{-\frac{1}{2}},0,0,0\right] \\ \mathbf{e}_{\hat{\phi}} &= \left[0,0,0,r^{-1}\right] \end{aligned}$$

The observed speed will be

$$\begin{aligned} V &= \frac{u^{\hat{\phi}}}{u^{\hat{t}}} = \frac{\mathbf{u}\cdot\mathbf{e}_{\hat{\phi}}}{-\mathbf{u}\cdot\mathbf{e}_{\hat{t}}} = \frac{r\,(d\phi/d\tau)}{(1-2M/r)^{\frac{1}{2}}(dt/d\tau)} \\ &= \frac{r}{(1-2M/r)^{\frac{1}{2}}}\frac{d\phi}{dt} = \frac{r}{(1-2M/r)^{\frac{1}{2}}}\left(\frac{M}{r^3}\right)^{\frac{1}{2}} \end{aligned}$$

where the last equality comes from (9.46). Evaluating this at $r = 3M$ we find

$$V = 1$$

the particle is moving at the speed of light!

b) Clearly this particle orbit coincides with the unstable photon orbit in the Schwarzschild geometry.